The hadronic vacuum polarisation contribution to the muon $g - 2$ from Lattice QCD

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Standard Model prediction — White Paper estimate

Contributions to the muon $g-2$ from electromagnetism, weak and strong interactions:

QED:	$116\,584\,718.9\,(1) \times 10^{-11}$	0.001 ppm	
Weak:	$153.6(1.0) \times 10^{-11}$	0.01 ppm	
Hadronic vacuum polarisation:	$6845(40) \times 10^{-11}$	0.34 ppm	$[0.6\%]$
Hadronic light-by-light scattering:	$92(18) \times 10^{-11}$	0.15 ppm	$[20\%]$

$$a_\mu^{\mathrm{SM}} = a_\mu^{\mathrm{QED}} + a_\mu^{\mathrm{weak}} + a_\mu^{\mathrm{hvp}} + a_\mu^{\mathrm{hlbl}} = 116\,591\,810(43) \times 10^{-11} \qquad 0.37\ \mathrm{ppm}$$

[Aoyama et al., Phys. Rep. 887 (2020) 1, arXiv:2006.04822]

This talk

QED and electroweak contributions to a_μ

Hadronic contributions in the White Paper and beyond

Hadronic vacuum polarisation: dispersion integrals versus lattice QCD

HVP in lattice QCD: recent calculations and cross checks

Assessment — Discussion — Outlook

QED contributions to a_μ

QED contribution has been worked out to in perturbation theory to 5-loop order:

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Complete Tenth-Order QED Contribution to the Muon $g - 2$

Tatsumi Aoyama,[1,2] Masashi Hayakawa,[3,2] Toichiro Kinoshita,[4,2] and Makiko Nio[2]

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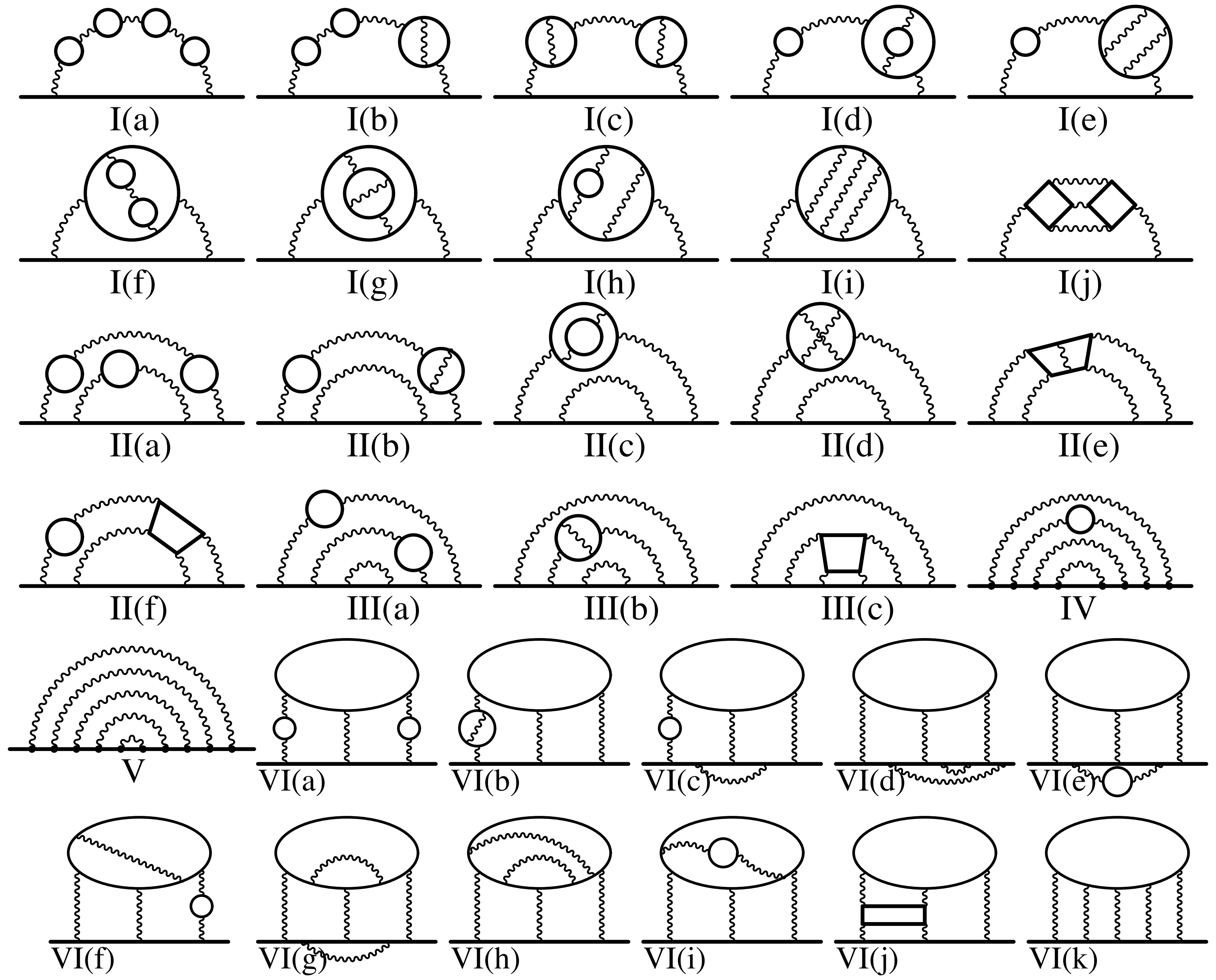
[4]*Laboratory for Elementary Particle Physics, Cornell University, Ithaca, New York 14853, USA*
(Received 24 May 2012; published 13 September 2012)

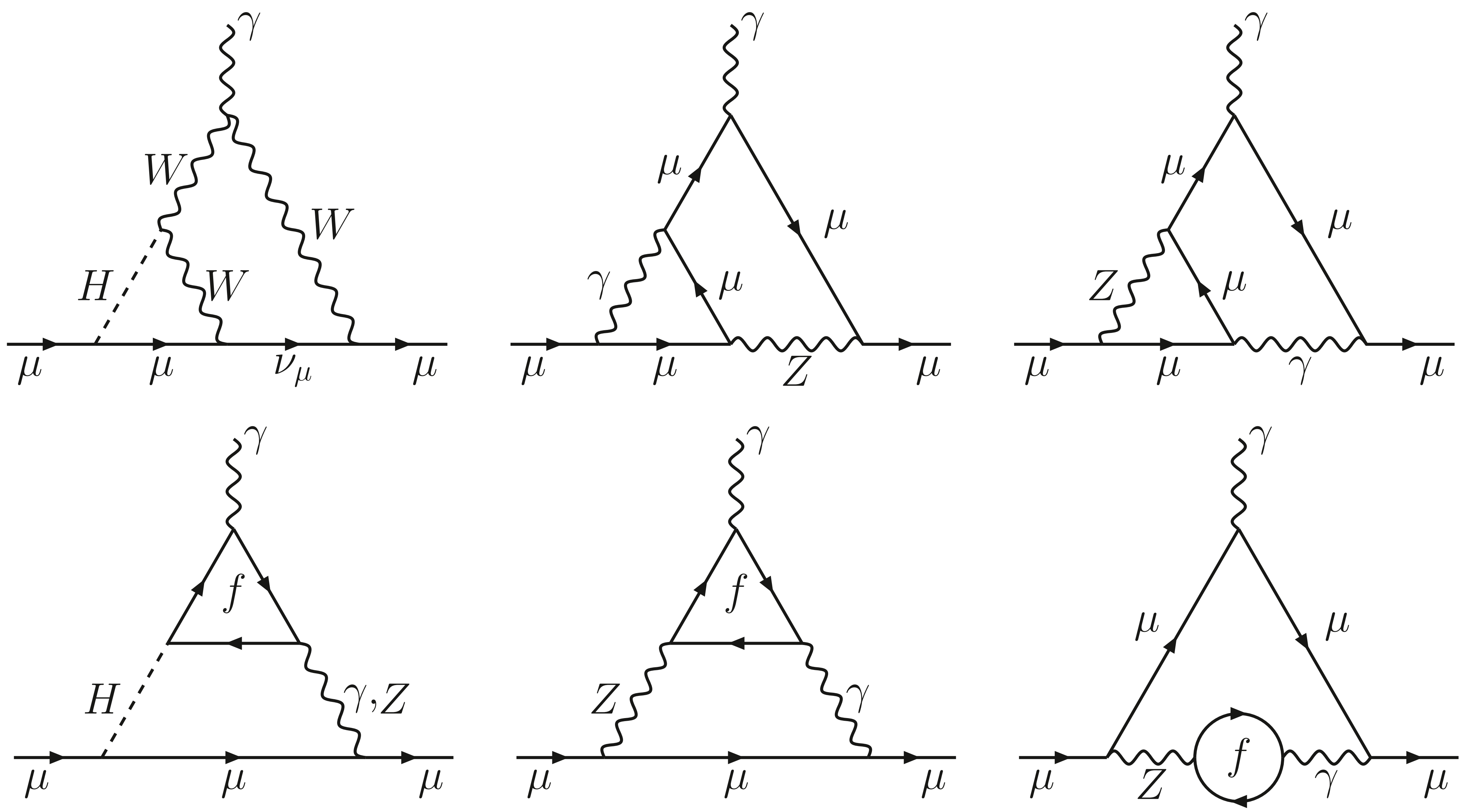
SM	116 591 810	100 %	#diagrams
QED(tot)	116 584 718.931	99,9939 %	
2	116 140 973.321	99,6133 %	1
4	413 217.626	0,3544 %	9
6	30 141.902	0,0259 %	72
8	381.004	0,0003 %	891
10	5.078	$4\cdot10^{-6}$ %	12672



Electroweak contributions to a_μ

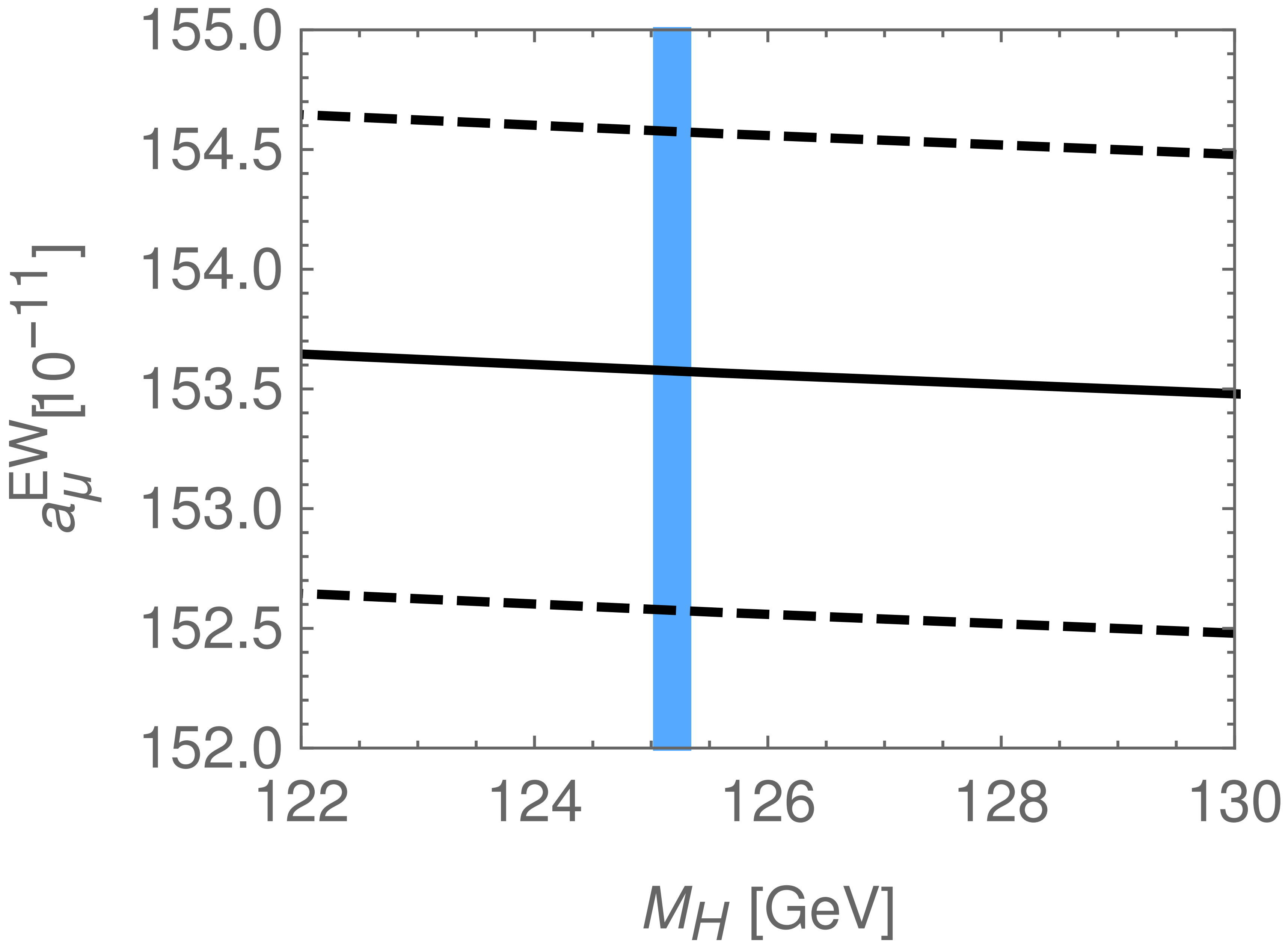
Weak contributions known to leading three-loop order

Sample two-loop diagrams:



$$a_\mu^{\mathrm{weak}} = (153.6 \pm 1.0) \times 10^{-11}$$

Dependence on the Higgs mass



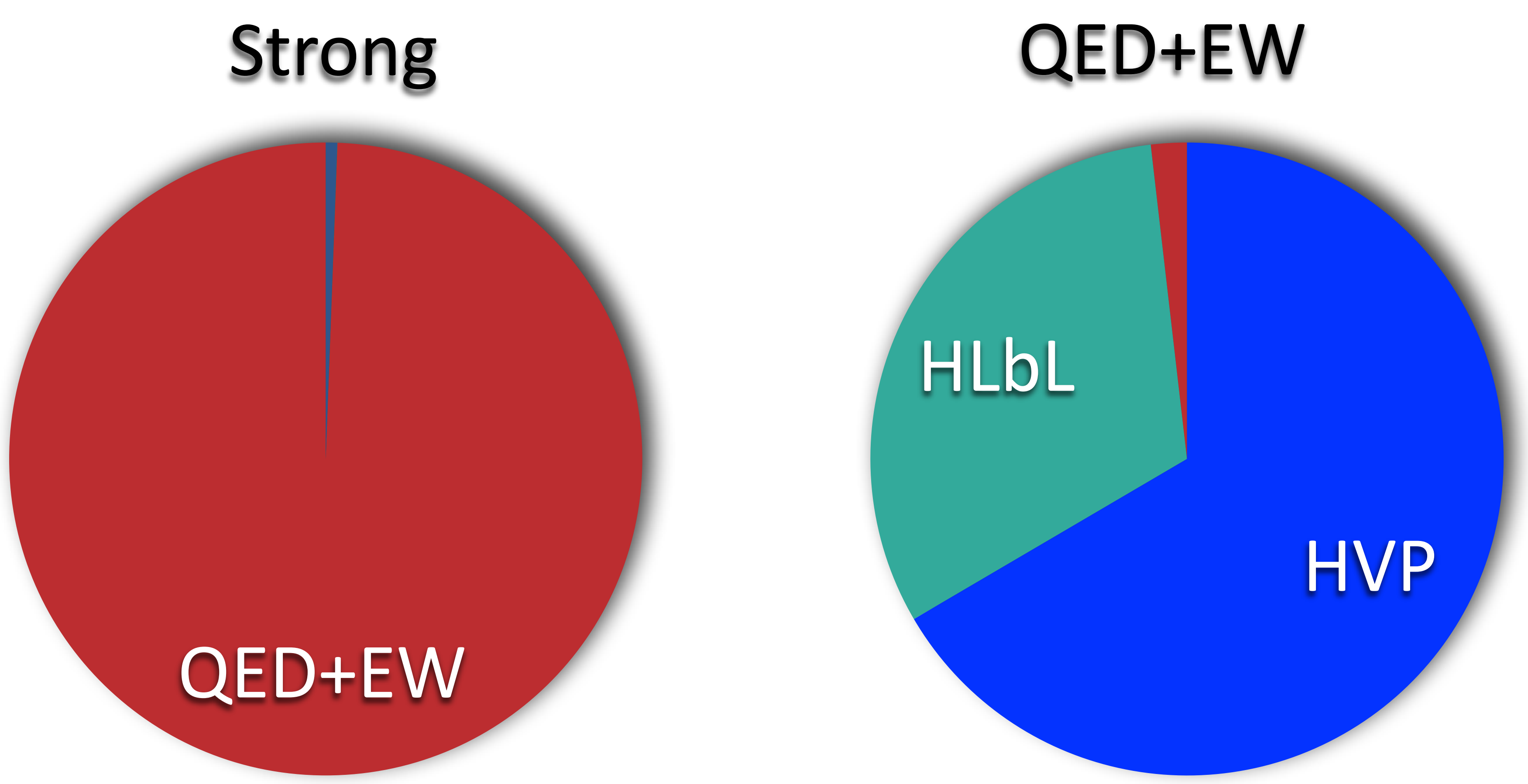
[Gnendiger et al., arXiv:1306.5546]

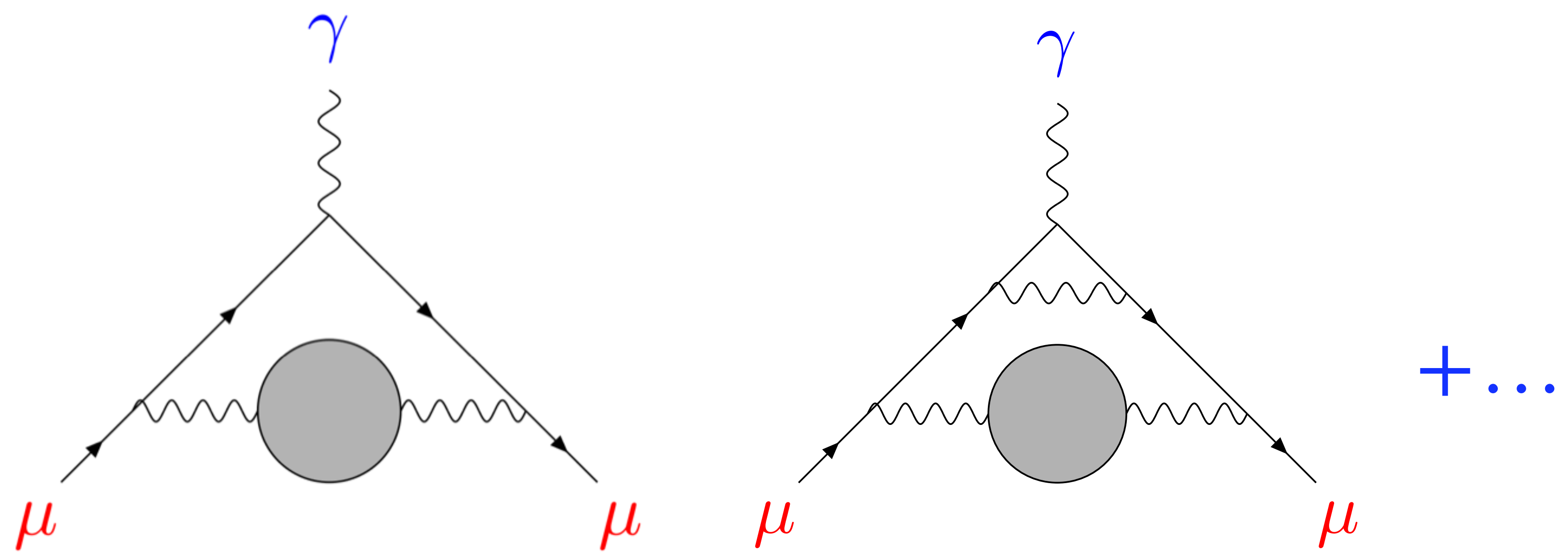
[Aoyama et al., Phys. Rep. 887 (2020) 1, arXiv:2006.04822]

Hadronic contributions to a_μ

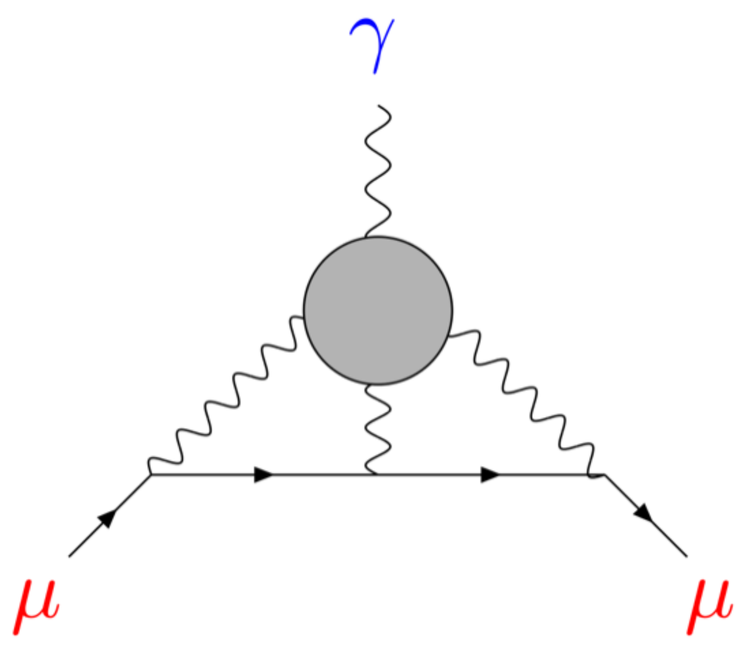
QED and electroweak contributions account for 99.994% of the SM prediction for a_μ

Error is dominated by strong interaction effects





Hadronic vacuum polarisation (HVP)

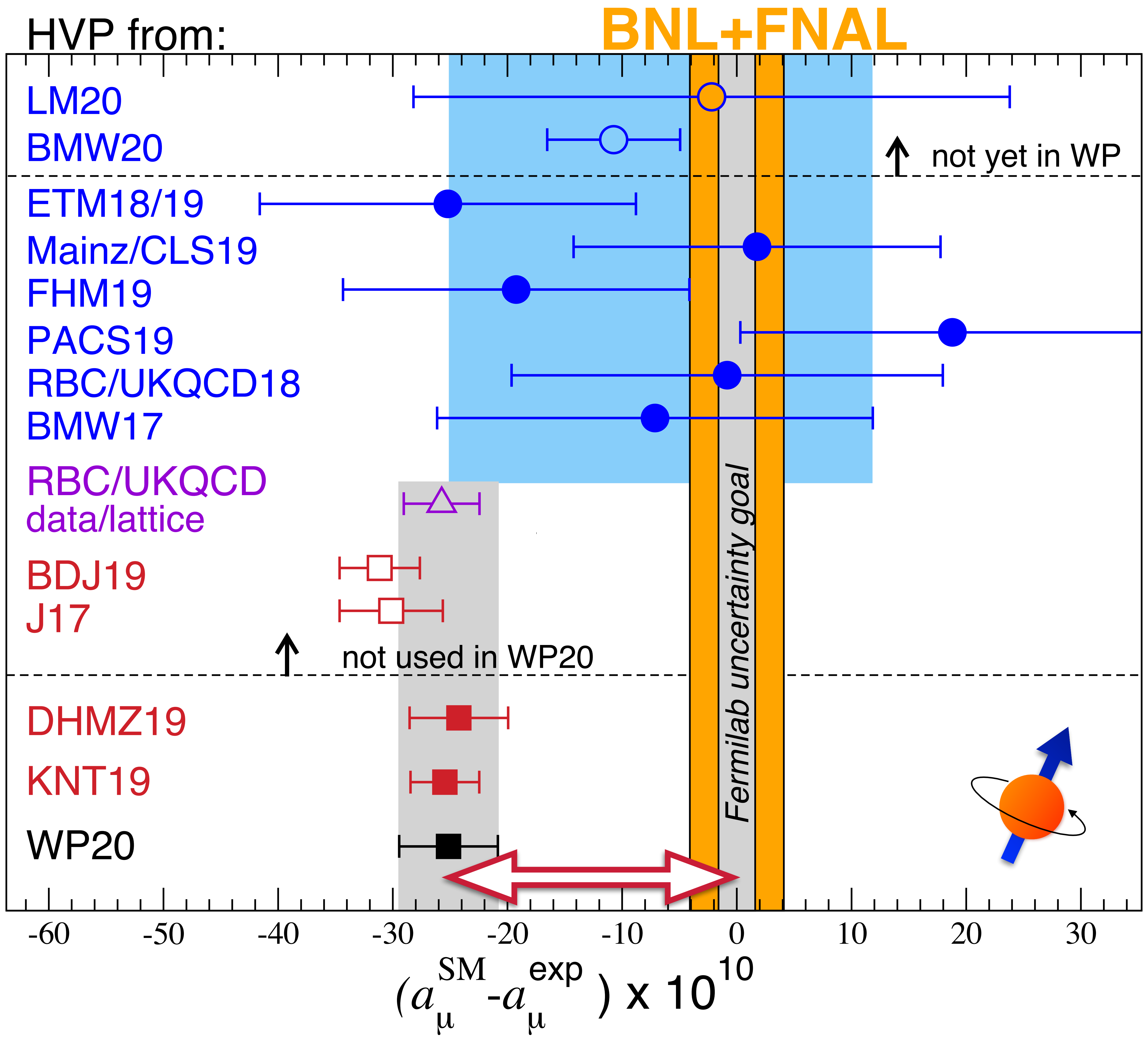


Hadronic light-by-light scattering (HLbL)

Two main approaches:

- Dispersion theory using experimentally determined cross sections ("data-driven")
- Lattice QCD calculations ("ab initio")

Standard Model prediction versus experiment



SM prediction (White Paper):

$$a_\mu^{\rm SM} = 116\,591\,810(43) \times 10^{-11}$$

FNAL E989 (2021):

$$a_\mu^{\rm E989} = 116\,592\,040(54) \times 10^{-11}$$

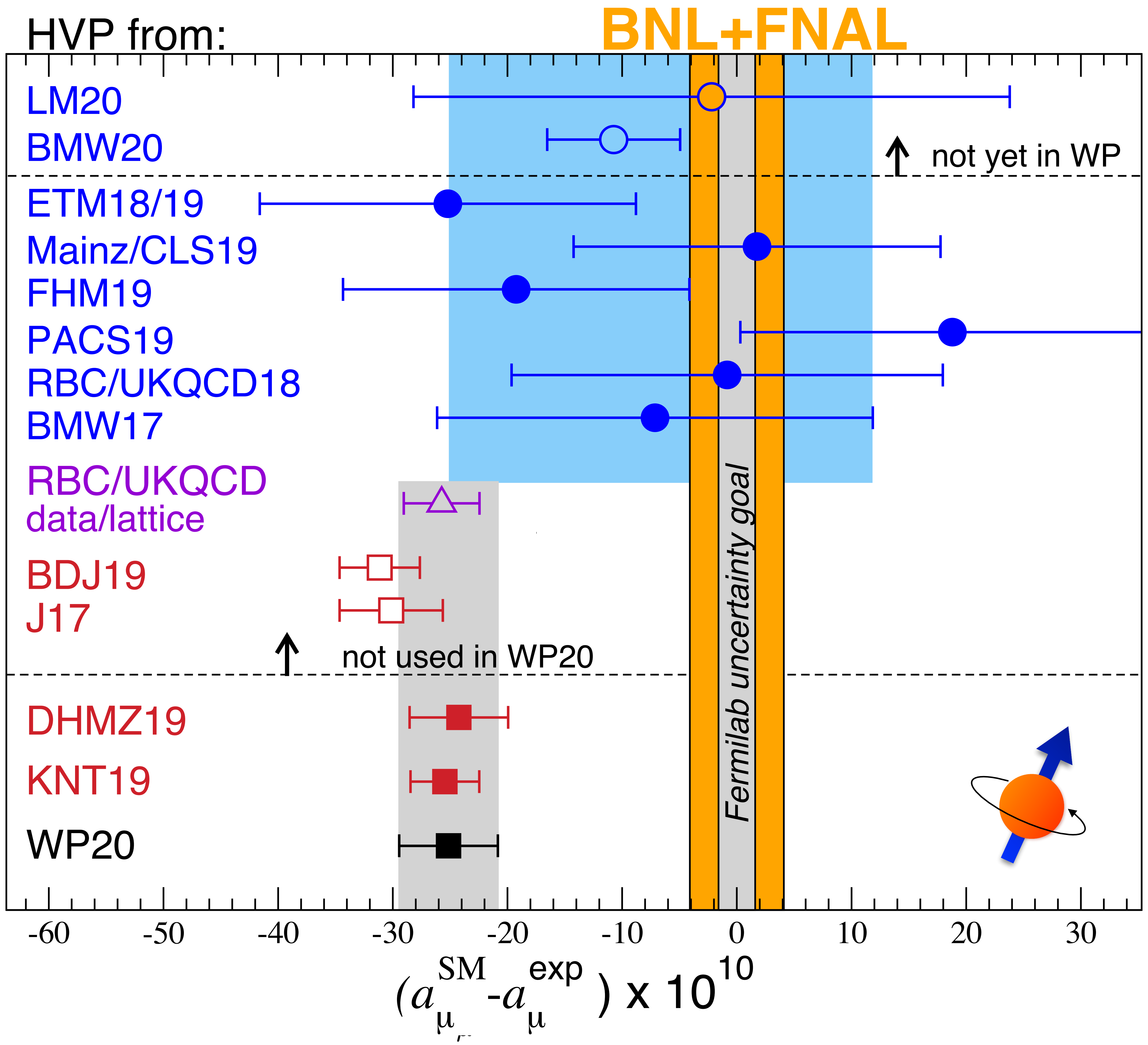
Combined with BNL E821 (2004):

$$a_\mu^{\rm exp} = 116\,592\,061(41) \times 10^{-11}$$

$$\Rightarrow \quad a_\mu^{\rm exp} - a_\mu^{\rm SM} = 251(59) \times 10^{-11} \quad (4.2\,\sigma)$$

[Aoyama et al., Phys. Rep. 887 (2020) 1; Colangelo et al., arXiv:2203.15810]

Hadronic vacuum polarisation: Data-driven approach versus lattice QCD



White Paper:

R-ratio: $a_\mu^{\text{hvp, LO}} = (693.1 \pm 4.0) \cdot 10^{-10}$ [0.6%]

LQCD: $a_\mu^{\text{hvp, LO}} = (711.6 \pm 18.4) \cdot 10^{-10}$ [2.6%]

Lattice QCD result by BMW Collab.:

$$a_\mu^{\text{hvp, LO}} = (707.5 \pm 2.3 \pm 5.0) \cdot 10^{-10} \quad [0.8\%]$$

[Borsányi et al., Nature 593 (2021) 7857, arXiv:2002.12347v3]

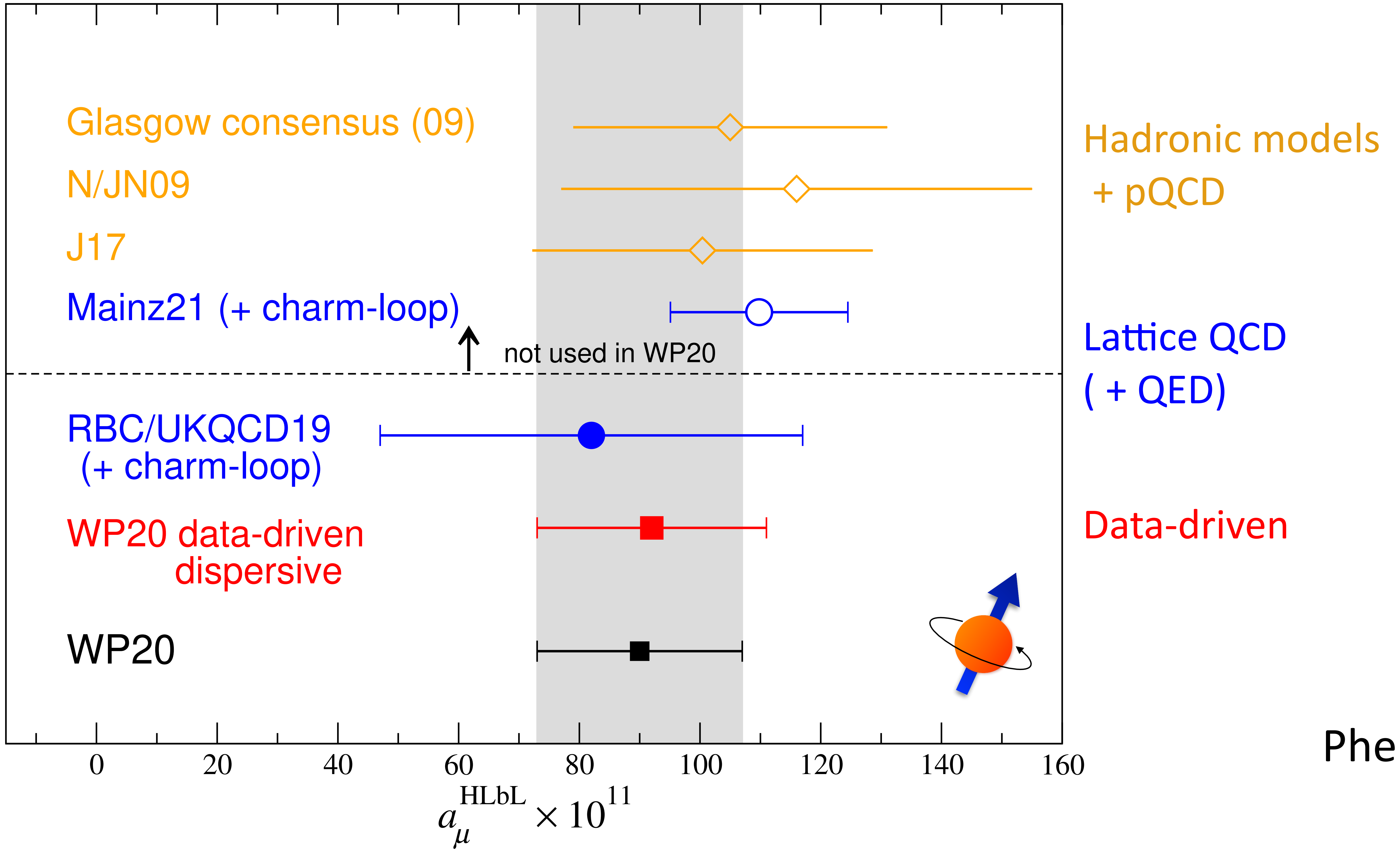
$$\Rightarrow \quad a_\mu^{\text{exp}} - a_\mu^{\text{SM}}\Big|_{\text{BMWc}}^{\text{hvp, LO}} = 107(70) \times 10^{-11} \quad (1.5\,\sigma)$$

(2.1σ tension with R-ratio)

Requires independent confirmation

Hadronic light-by-light scattering

[Aoyama et al., Phys. Rep. 887 (2020) 1; Colangelo et al., arXiv:2203.15810]



White Paper:

$$a_\mu^{\rm hlbl} = (92 \pm 18) \cdot 10^{-11}$$

Direct lattice calculations (Mainz):

$$a_\mu^{\rm hlbl} = (106.8 \pm 14.7) \cdot 10^{-11}$$

(excluding charm loop)

$$a_\mu^{\rm hlbl,\,c} = (2.8 \pm 0.5) \cdot 10^{-11}$$

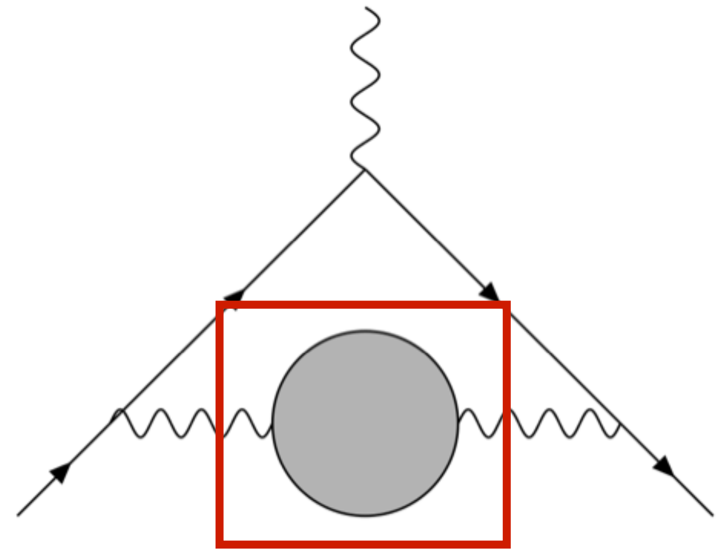
[Chao et al., Eur. Phys. J. C81 (2021) 7, 651; arXiv:2204.08844]

Phenomenological estimate: $a_\mu^{\rm hlbl,\,c} = (3 \pm 1) \cdot 10^{-11}$

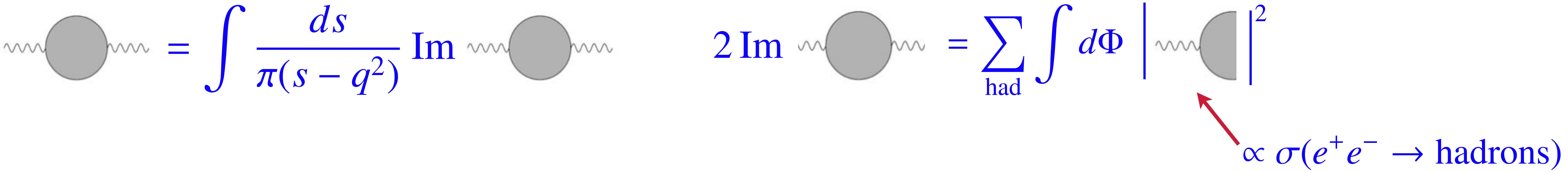
$$\Rightarrow \quad a_\mu^{\rm exp} - a_\mu^{\rm SM}\Big|_{\rm Mainz}^{\rm hlbl} = 234(59) \times 10^{-11} \quad (4.0\,\sigma)$$

Hadronic light-by-light scattering not the dominant source of uncertainty!

Hadronic vacuum polarisation from dispersion theory



Analyticity, unitarity & optical theorem imply:



$$a_\mu^{\rm hvp} = \left(\frac{\alpha m_\mu}{3\pi}\right)^2 \int_{m_{\pi^0}^2}^{\infty} ds\, \frac{R_{\rm had}(s)\hat{K}(s)}{s^2}, \qquad R_{\rm had}(s) = \frac{3s}{4\pi\,\alpha(s)}\,\sigma(e^+e^- \to \text{hadrons})$$

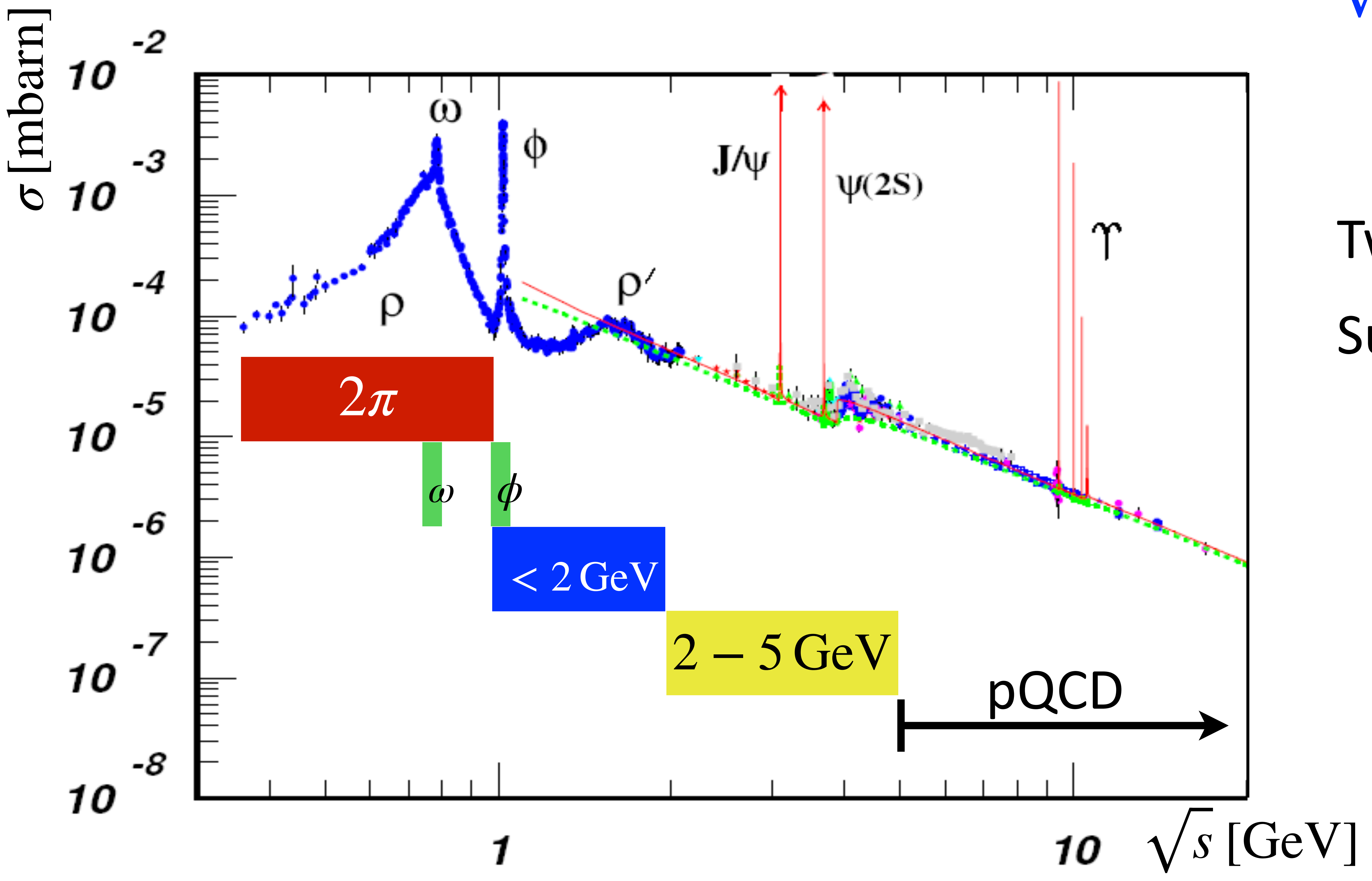
"R-ratio"

Hadronic effects cannot be treated in perturbation theory

- Use experimental data for $R_{\rm had}(s)$ in the low-energy regime ("data-driven approach")
- Standard Model prediction is subject to experimental uncertainties

Data-driven approach: Hadronic cross sections

$$a_\mu^{\rm hvp} = \left(\frac{\alpha m_\mu}{3\pi}\right)^2 \int_{m_{\pi^0}^2}^{\infty} ds\, \frac{R_{\rm had}(s)\hat{K}(s)}{s^2}$$



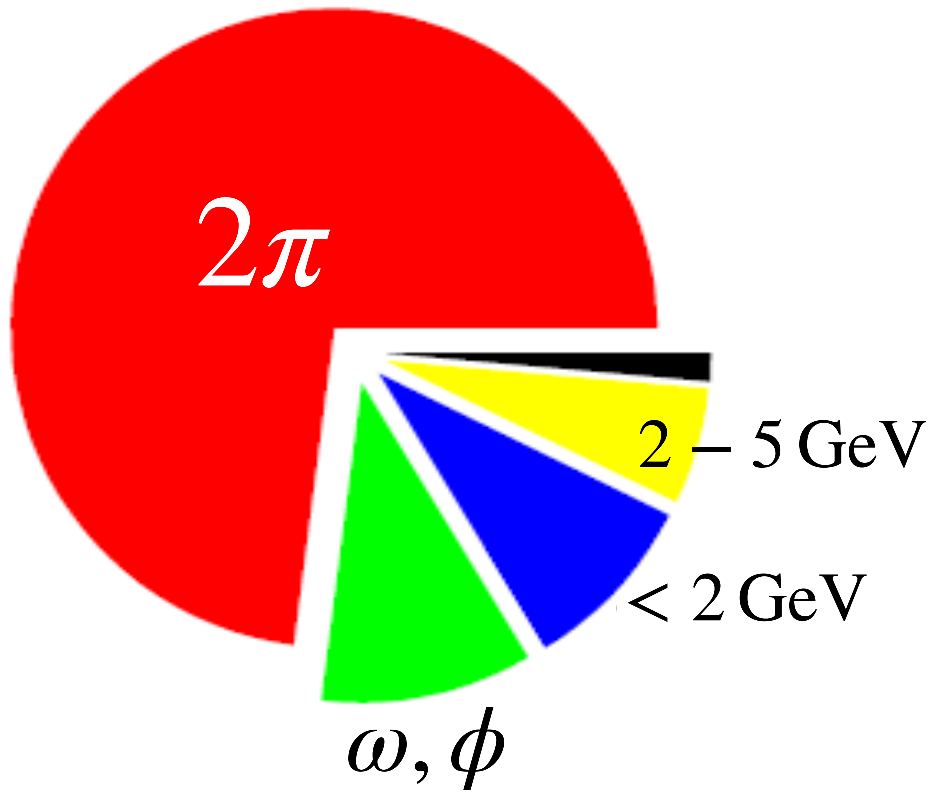
Decade-long effort to measure e^+e^- cross sections

$\sqrt{s} \lesssim 2\,\mathrm{GeV}$: sum of exclusive channels

$\sqrt{s} > 2\,\mathrm{GeV}$: inclusive channels, narrow resonances, perturbative QCD

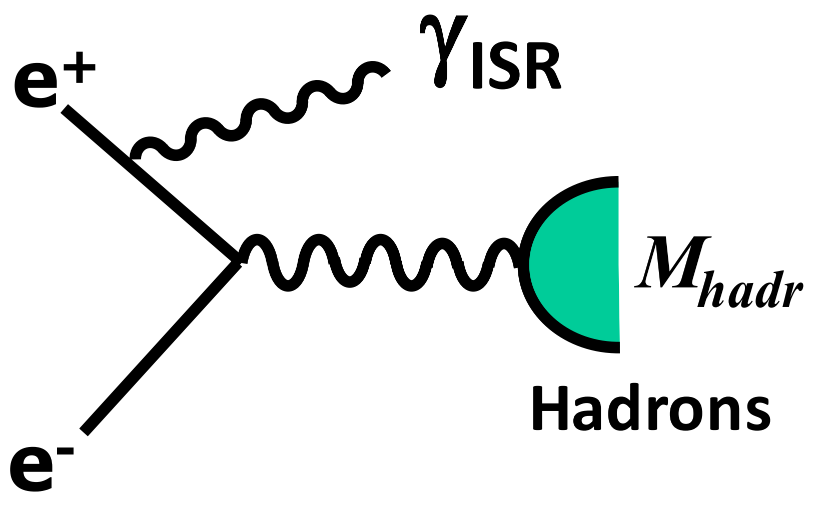
Two-pion channel accounts for $\approx 70\,\%$ of LO-HVP

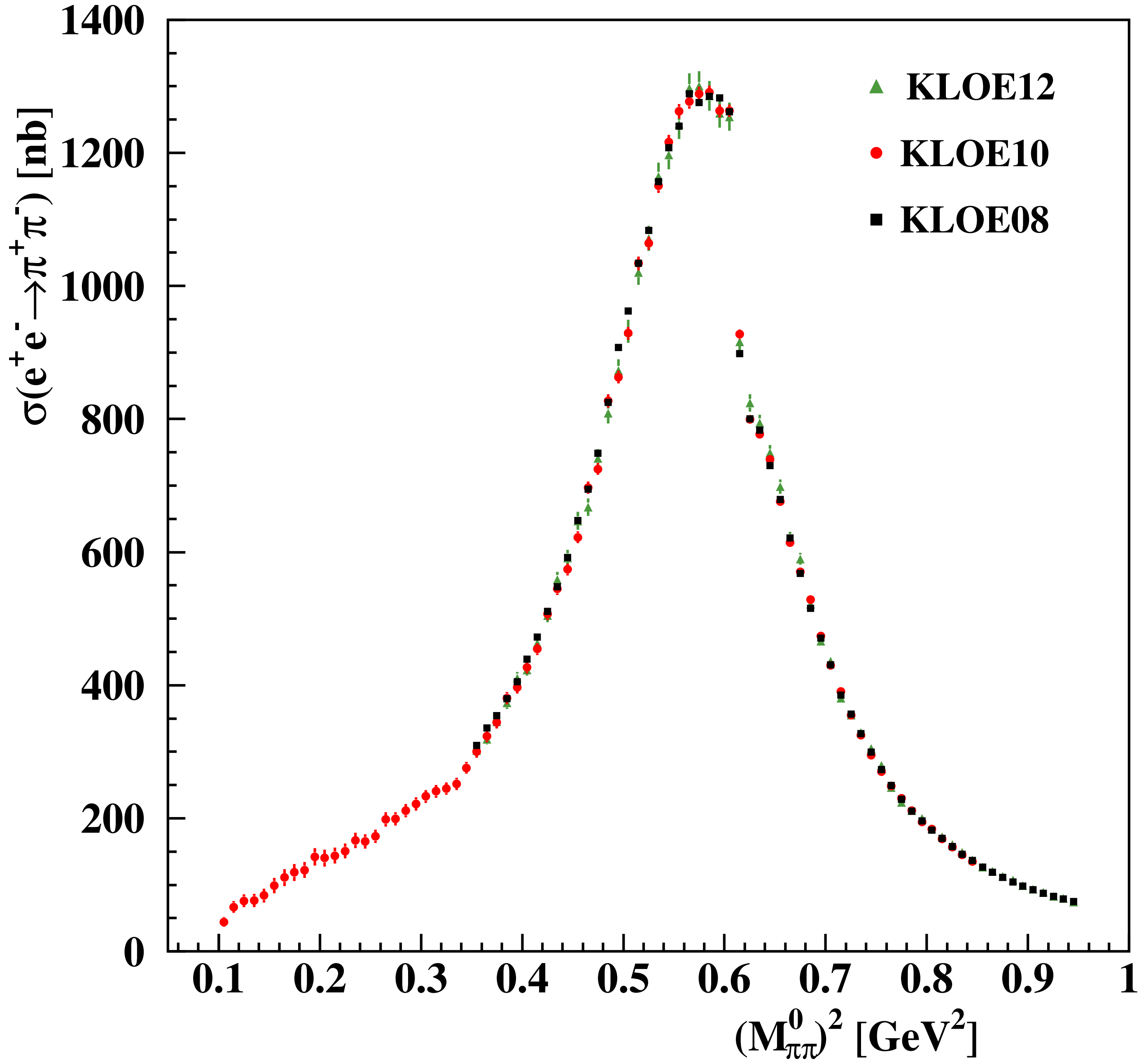
Subleading channels: ω, ϕ decays, final states with 3 pions, 2 kaons, 4 pions,…

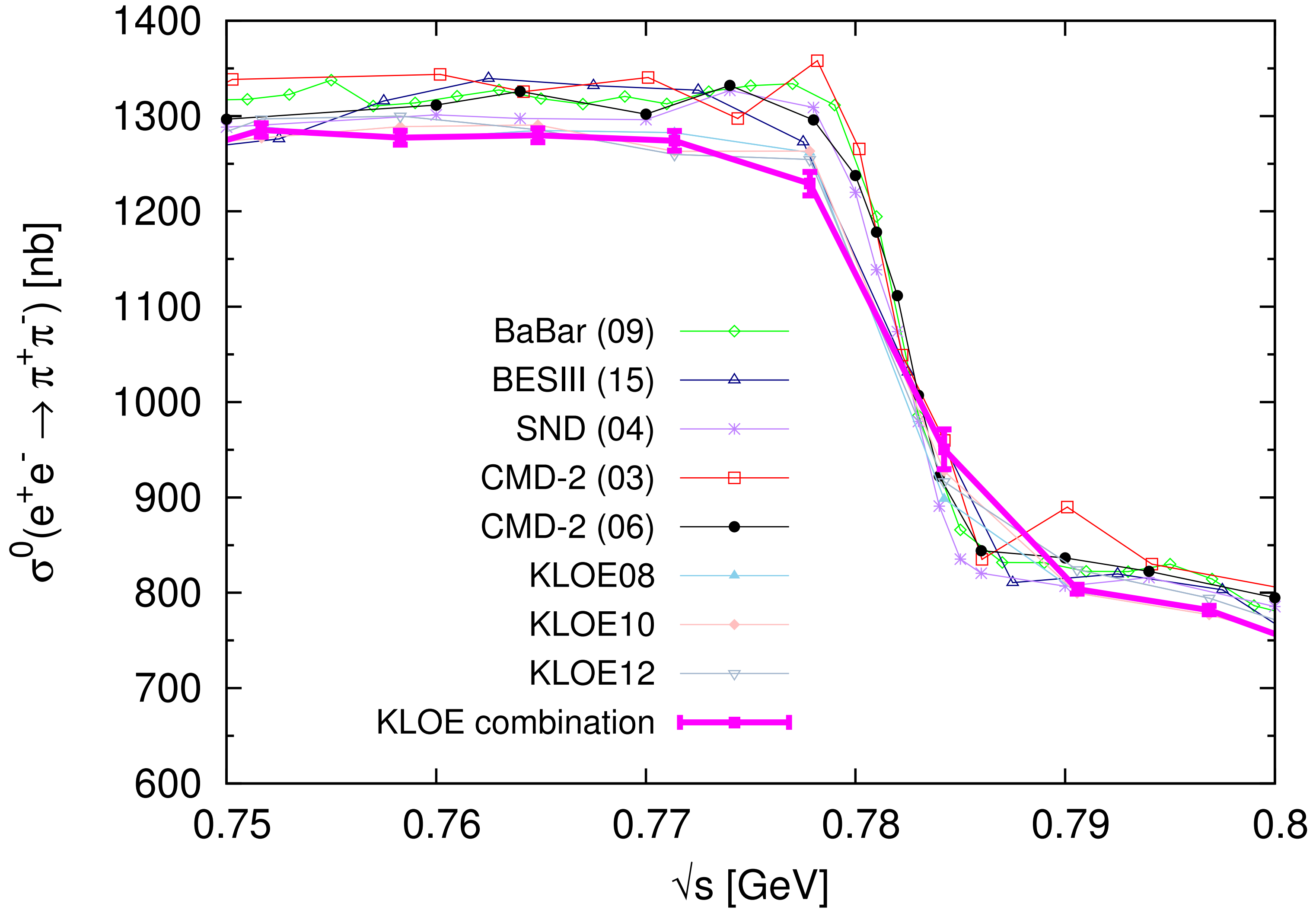


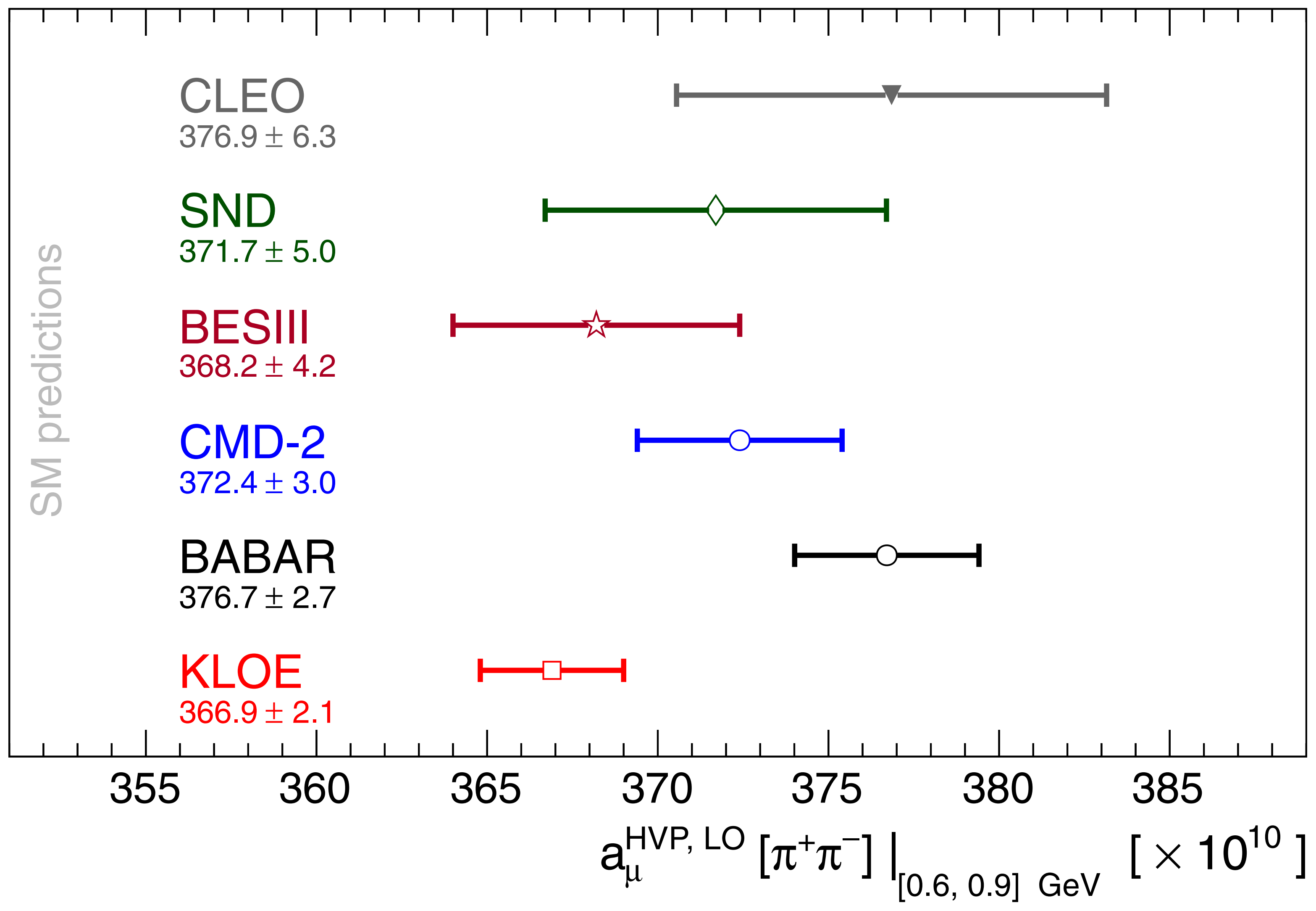
Two-pion channel

Initial State Radiation technique: energy scan at fixed collider energy (BaBar, KLOE, BESIII)









- Tension in the ISR data for $e^+e^- \to \pi^+\pi^-$ between BaBar and KLOE
- Extended analysis of BaBar data in progress
- New data: SND-3 (published) and CMD-3 (expected)
- Future prospects at BESIII, Belle II

Evaluation of the dispersion integral

Many different groups and analyses (DHMZ, KNT, FJ, CHHKS, BHLS,…)

Disagreement for some exclusive channels

	DHMZ19	KNT19	Difference
$\pi^+\pi^-$	507.85(0.83)(3.23)(0.55)	504.23(1.90)	3.62
$\pi^+\pi^-\pi^0$	46.21(0.40)(1.10)(0.86)	46.63(94)	−0.42
$\pi^+\pi^-\pi^+\pi^-$	13.68(0.03)(0.27)(0.14)	13.99(19)	−0.31
$\pi^+\pi^-\pi^0\pi^0$	18.03(0.06)(0.48)(0.26)	18.15(74)	−0.12
K^+K^-	23.08(0.20)(0.33)(0.21)	23.00(22)	0.08
K_SK_L	12.82(0.06)(0.18)(0.15)	13.04(19)	−0.22
$\pi^0\gamma$	4.41(0.06)(0.04)(0.07)	4.58(10)	−0.17
Sum of the above	626.08(0.95)(3.48)(1.47)	623.62(2.27)	2.46
$[1.8, 3.7]$ GeV (without $c\bar{c}$)	33.45(71)	34.45(56)	−1.00
J/ψ, $\psi(2S)$	7.76(12)	7.84(19)	−0.08
$[3.7, \infty)$ GeV	17.15(31)	16.95(19)	0.20
Total $a_\mu^{\mathrm{HVP,\ LO}}$	$694.0(1.0)(3.5)(1.6)(0.1)_\psi(0.7)_{\mathrm{DV+QCD}}$	692.8(2.4)	1.2

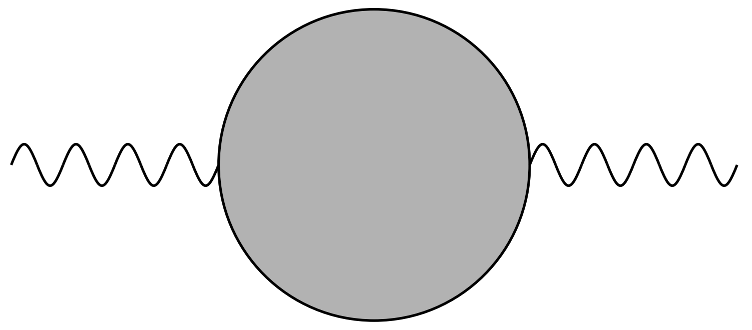
Merging procedure: average of individual results + theoretical constraints + conservative error estimate (reflecting tensions in the data, differences in procedures)

$$a_\mu^{\mathrm{hvp,\,LO}} = 693.1(2.8)_{\mathrm{exp}}(2.8)_{\mathrm{syst}}(0.7)_{\mathrm{DV+QCD}} \times 10^{-10} = 693.1(4.0) \times 10^{-10} \qquad [0.6\%]$$

Hadronic vacuum polarisation from Lattice QCD

Lattice QCD does NOT determine the R-ratio from first principles

Primary observable: $G(t) = -\frac{a^3}{3}\sum_{k=1}^{3}\sum_{\vec{x}} G_{kk}(\vec{x},t), \quad G_{\mu\nu}(x) = \left\langle j_\mu^{\rm em}(x) j_\nu^{\rm em}(0)\right\rangle$



Electromagnetic current: $j_\mu^{\rm em}(x) = \frac{2}{3}\bar{u}\gamma_\mu u - \frac{1}{3}\bar{d}\gamma_\mu d - \frac{1}{3}\bar{s}\gamma_\mu s + \frac{2}{3}\bar{c}\gamma_\mu c + \ldots$

Vacuum polarisation function for Euclidean momenta:

$$\underbrace{4\pi^2\left\{\Pi(-q^2)-\Pi(0)\right\}}_{\hat{\Pi}(Q^2)} = \frac{Q^2}{3}\int_0^\infty ds\,\frac{R(s)}{s(s+Q^2)} = \frac{1}{Q^2}\int_0^\infty dt\,G(t)\left[Q^2t^2 - 4\sin^2\left(\tfrac{1}{2}Qt\right)\right]$$

t : Euclidean time *[Bernecker & Meyer 2011]*

$$a_\mu^{\rm hvp,LO} = \left(\frac{\alpha}{\pi}\right)^2\int_0^\infty dQ^2\, f(Q^2)\,\hat{\Pi}(Q^2) = \left(\frac{\alpha}{\pi}\right)^2\int_0^\infty dt\; G(t)\;\underbrace{\int_0^\infty dQ^2\, f(Q^2)\left[t^2 - \frac{4}{Q^2}\sin^2\left(\tfrac{1}{2}Qt\right)\right]}_{\tilde{K}(t)}$$

[Lautrup, Peterman & de Rafael 1972, Blum 2002]

Hadronic vacuum polarisation from Lattice QCD

Time-momentum representation (TMR) *[Bernecker & Meyer 2011]*

$$a_\mu^{\mathrm{hvp,LO}} = \left(\frac{\alpha}{\pi}\right)^2 \int_0^\infty dt\, \tilde{K}(t)\, G(t), \qquad G(t) = -\frac{a^3}{3} \sum_k \sum_{\vec{x}} \left\langle j_k^{\mathrm{em}}(\vec{x},t)\, j_k^{\mathrm{em}}(0) \right\rangle$$

$\tilde{K}(t)$: analytically known kernel function

- No reliance on experimental data, except for simple hadronic quantities such as $m_{\mathrm{nucl}}, m_K, \ldots$ $\rightarrow$ scale setting, calibration
- Not sensitive to exclusive hadronic channels
- Can perform an explicit quark flavour separation of $a_\mu^{\mathrm{hvp,LO}}$

Why computing the HVP contribution is a challenge

$$a_\mu^{\mathrm{hvp,LO}} = \left(\frac{\alpha}{\pi}\right)^2 \int_0^\infty dt\, \tilde{K}(t)\, G(t), \quad G(t) = -\frac{a^3}{3} \sum_k \sum_{\vec{x}} \left\langle j_k^{\mathrm{em}}(\vec{x},t)\, j_k^{\mathrm{em}}(0) \right\rangle$$

Sub-percent statistical precision;

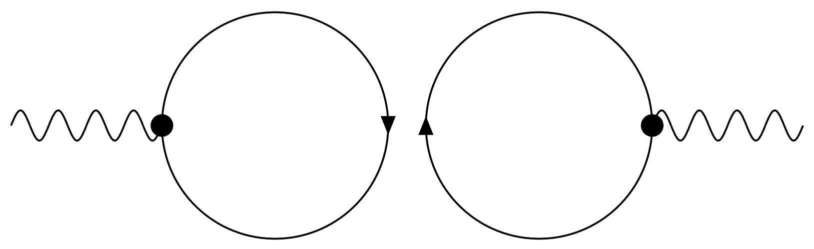
exponentially growing signal-to-noise in $G(t)$ as $t \to \infty$

Correct for finite-volume effects

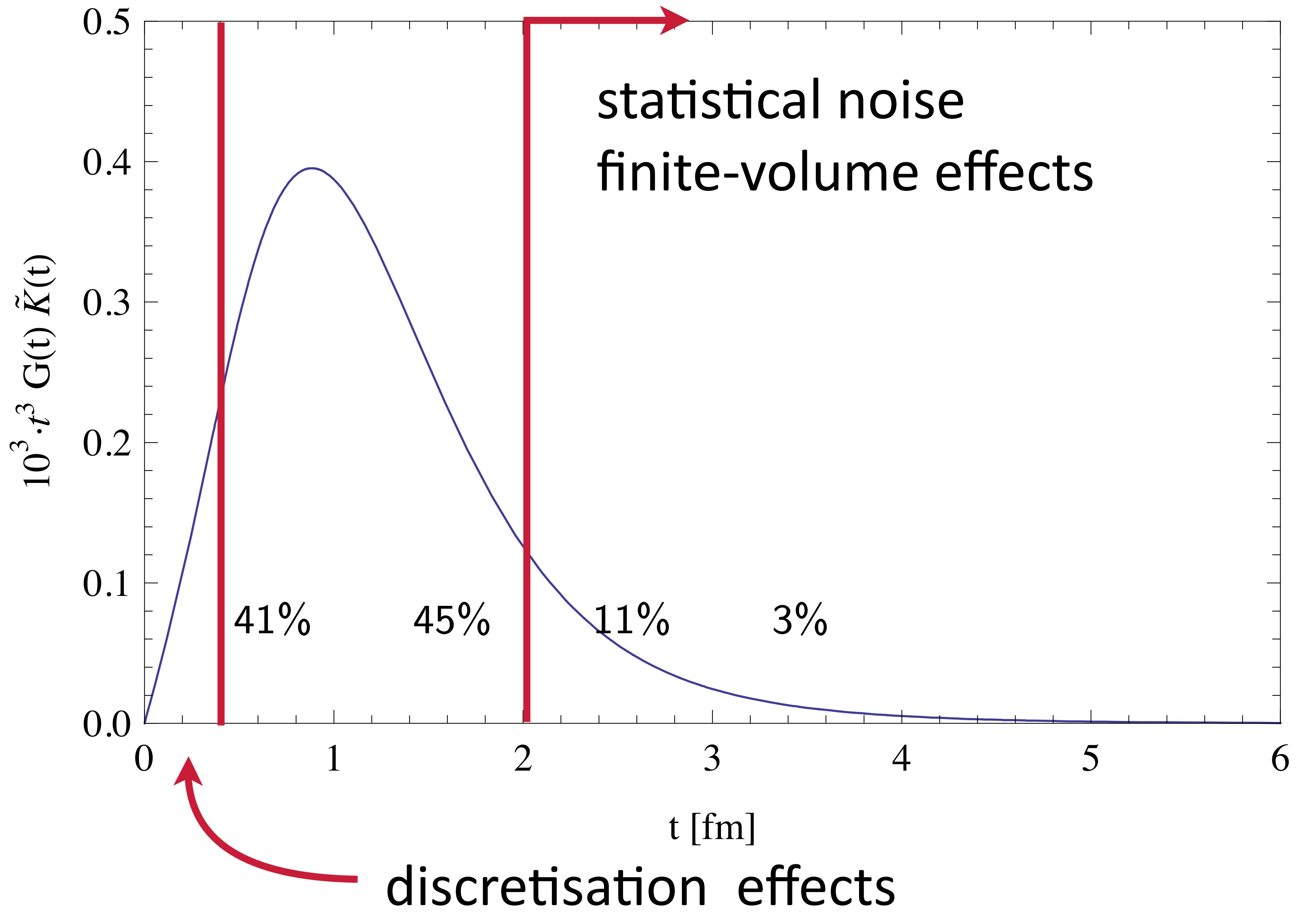
Control discretisation effects

Quark-disconnected diagrams:

control statistical & stochastic noise



Isospin breaking: $m_u \neq m_d$ and QED



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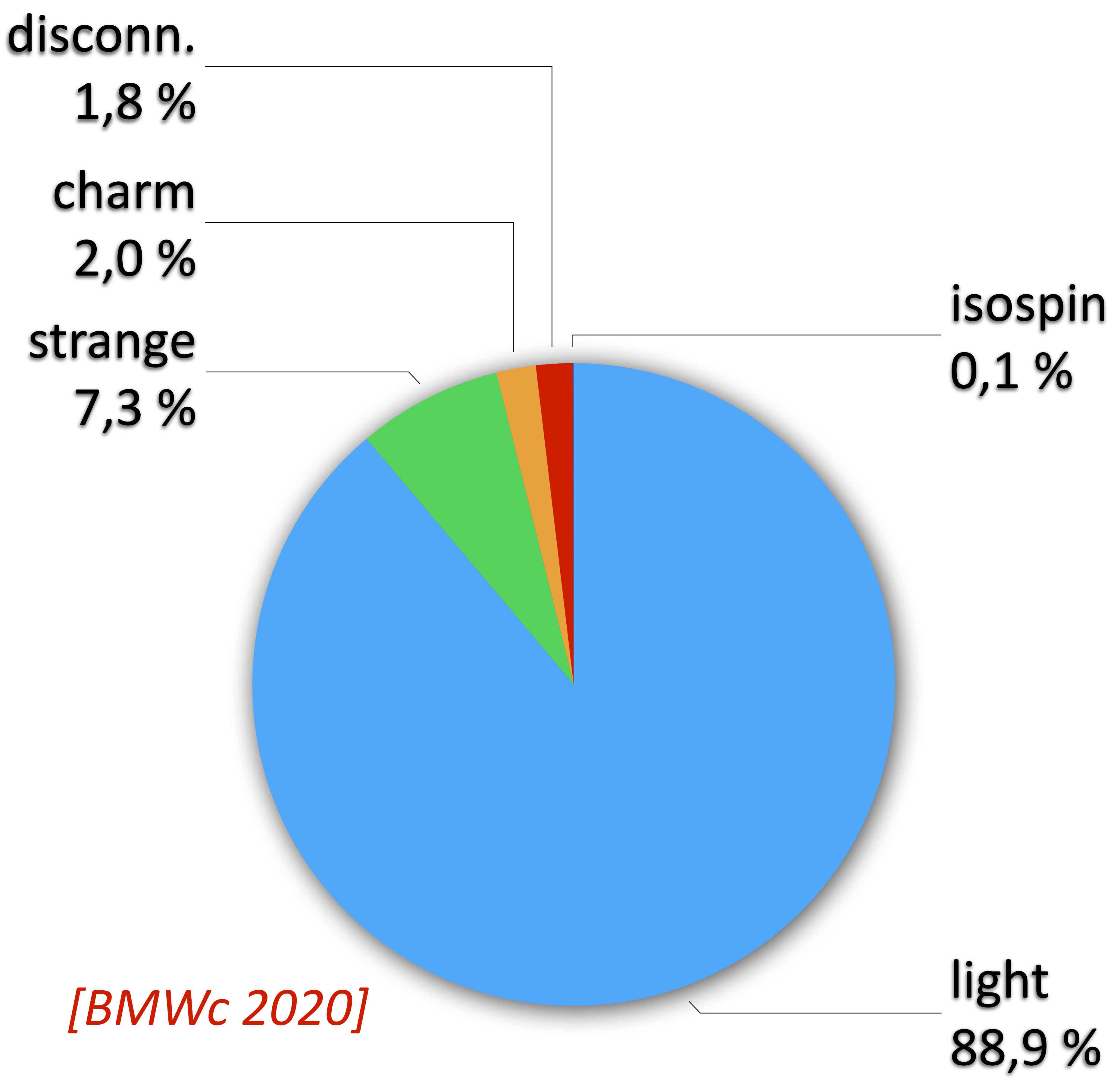
Isospin breaking: $m_u \neq m_d$ and QED

Light-quark connected contribution dominates

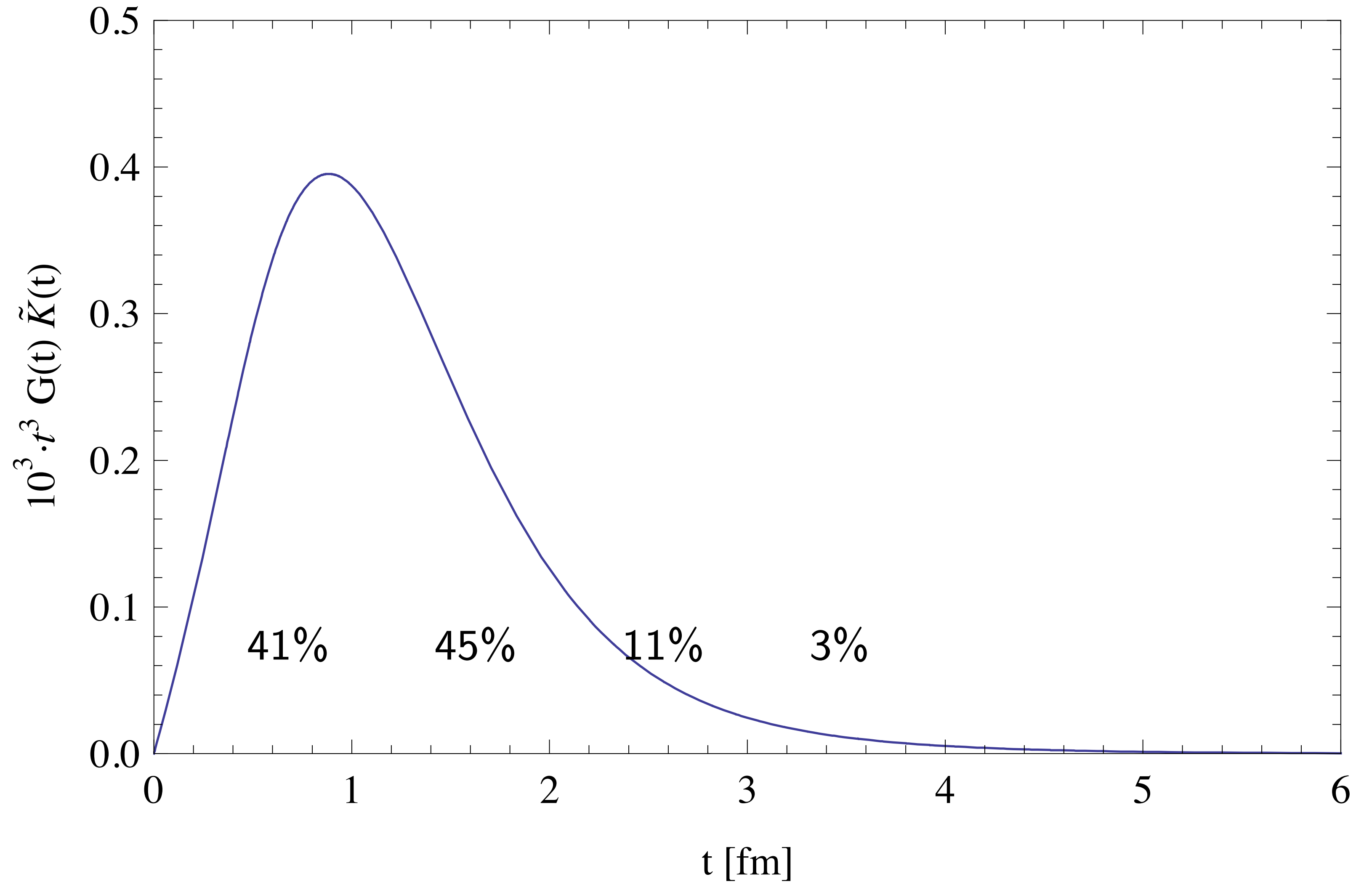


[BMWc 2020]

Controlling the long-distance tail of $G(t)$

- Long-distance tail of the light quark contribution to $G(t)$: limiting factor for overall statistical precision

- Correlator dominated by isovector two-pion contribution



Strategies:

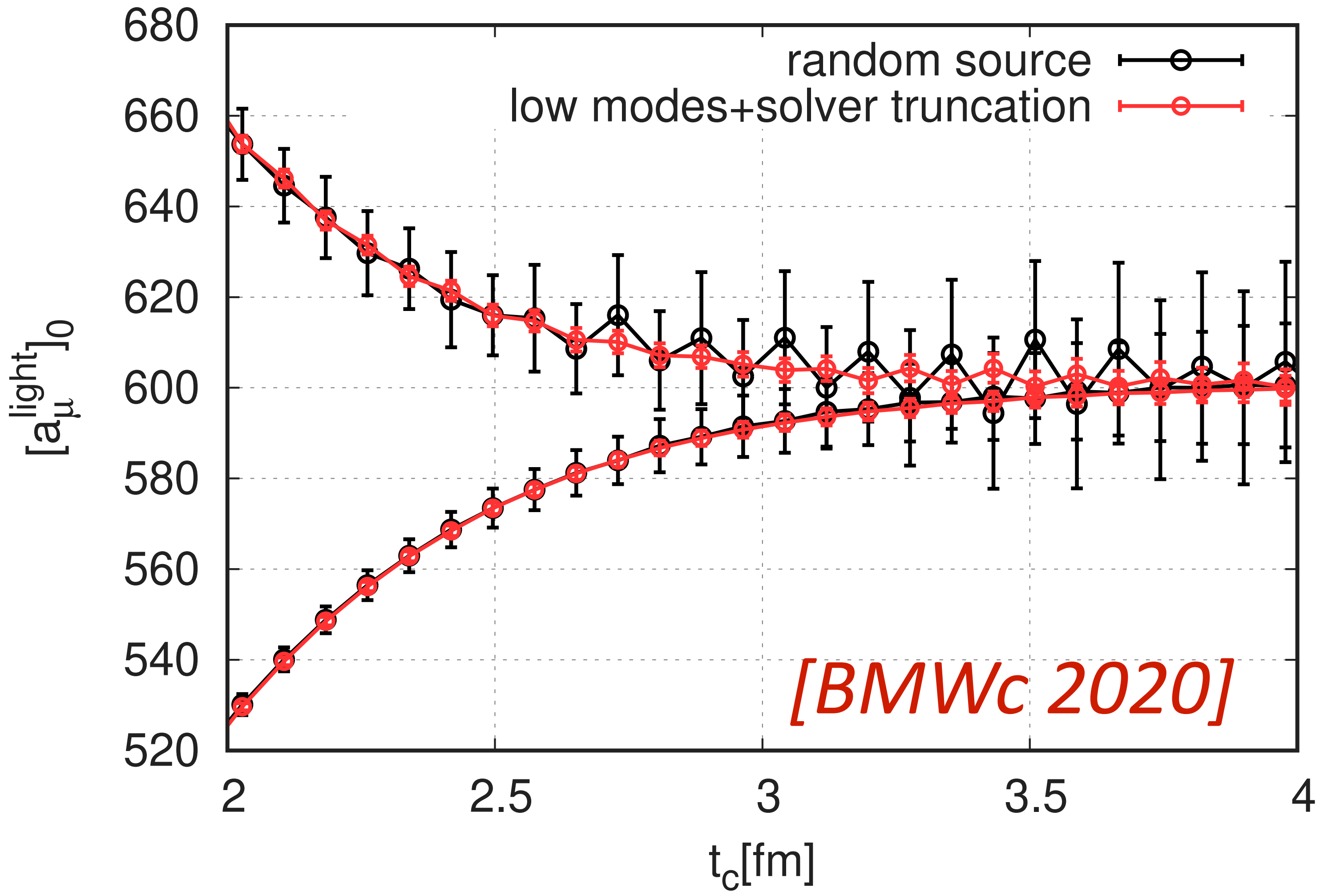
- Dedicated calculations of the spectrum in isovector channel and/or pion form factor $F_\pi(\omega)$

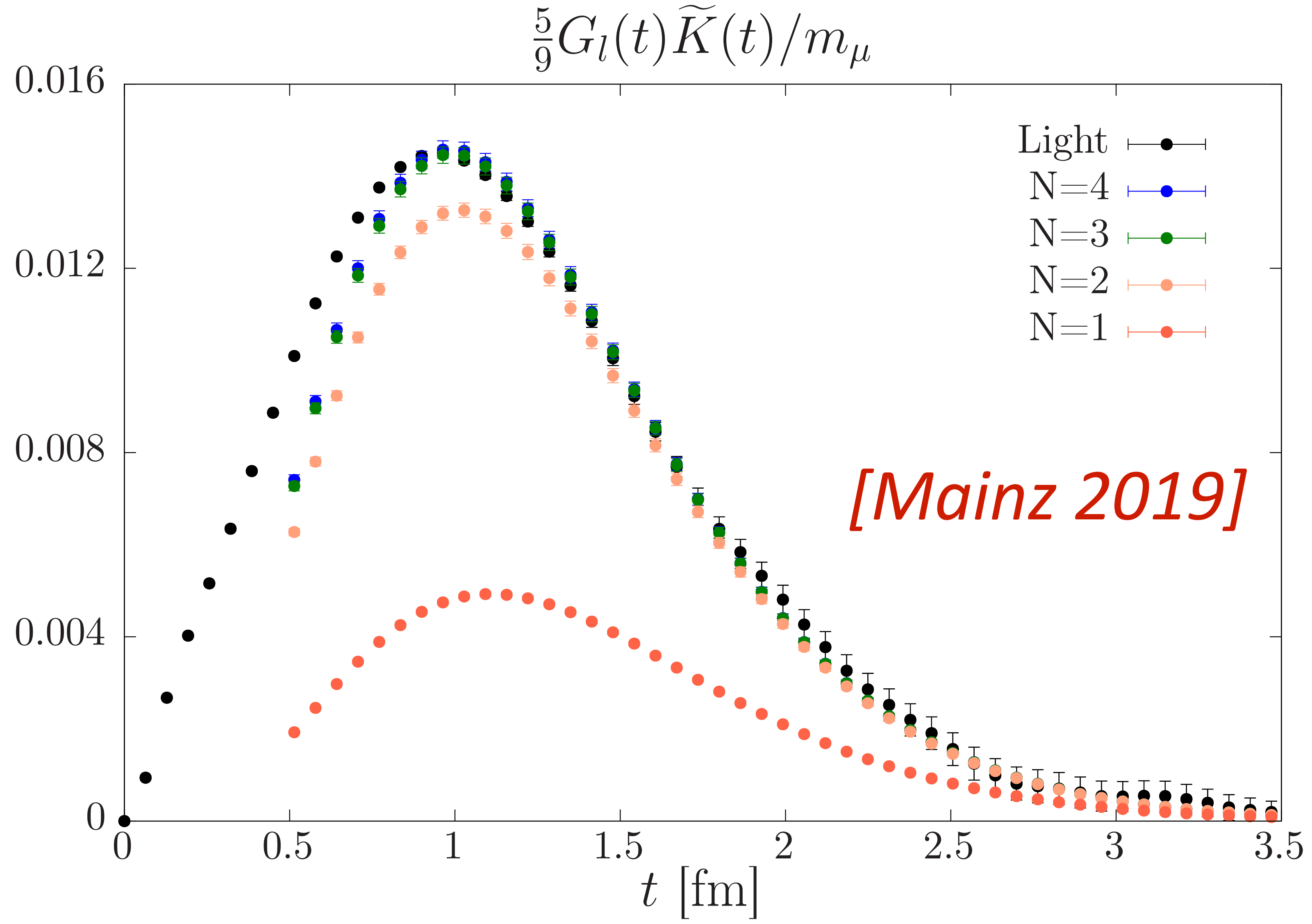
- "Bounding method":

$$0 \leq G(t) \leq G(t_c)\,\frac{G^{\pi\pi}(t)}{G^{\pi\pi}(t_c)}$$

- Noise-reduction methods: AMA, LMA, truncated solver

(can be combined)



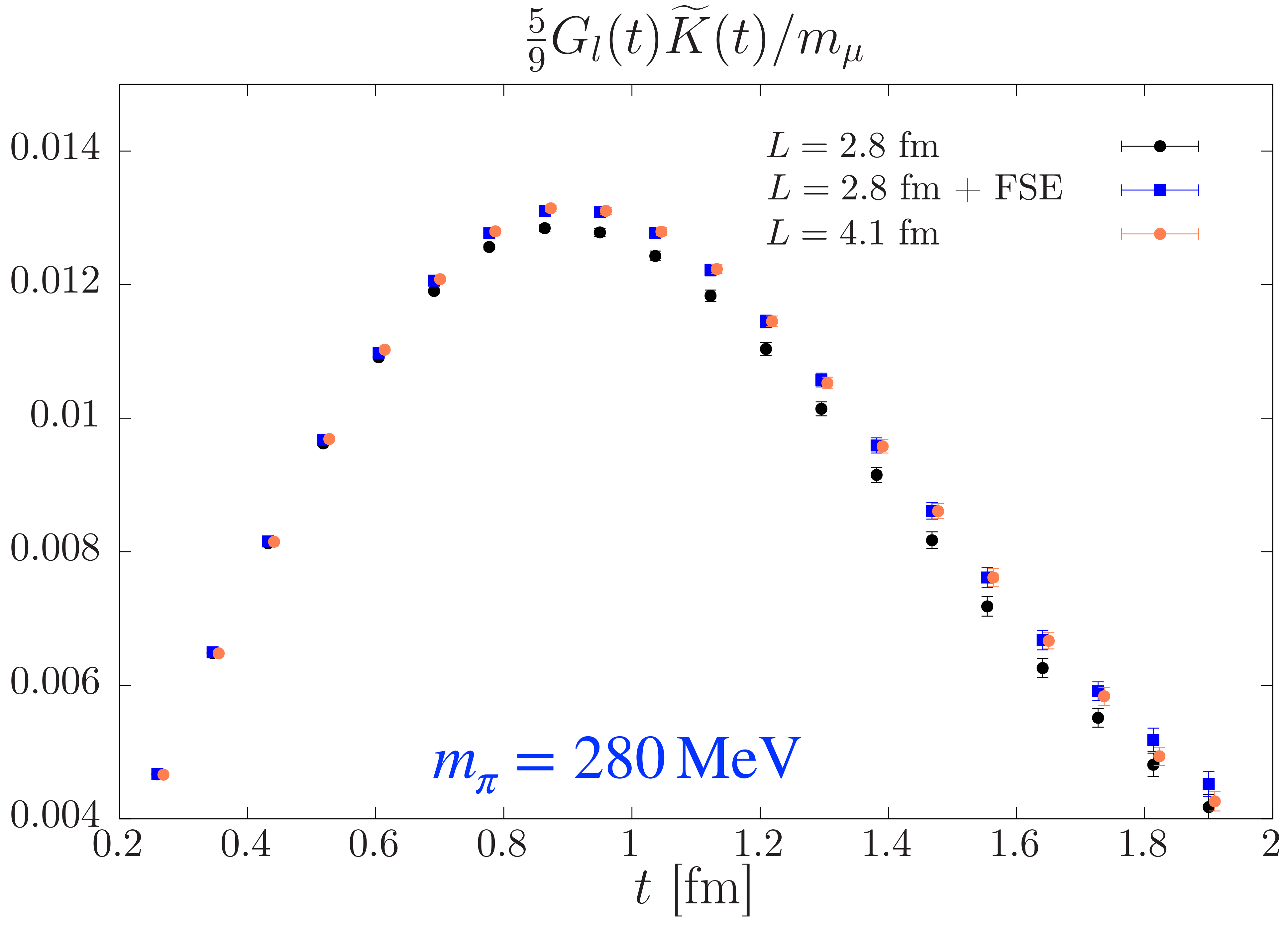


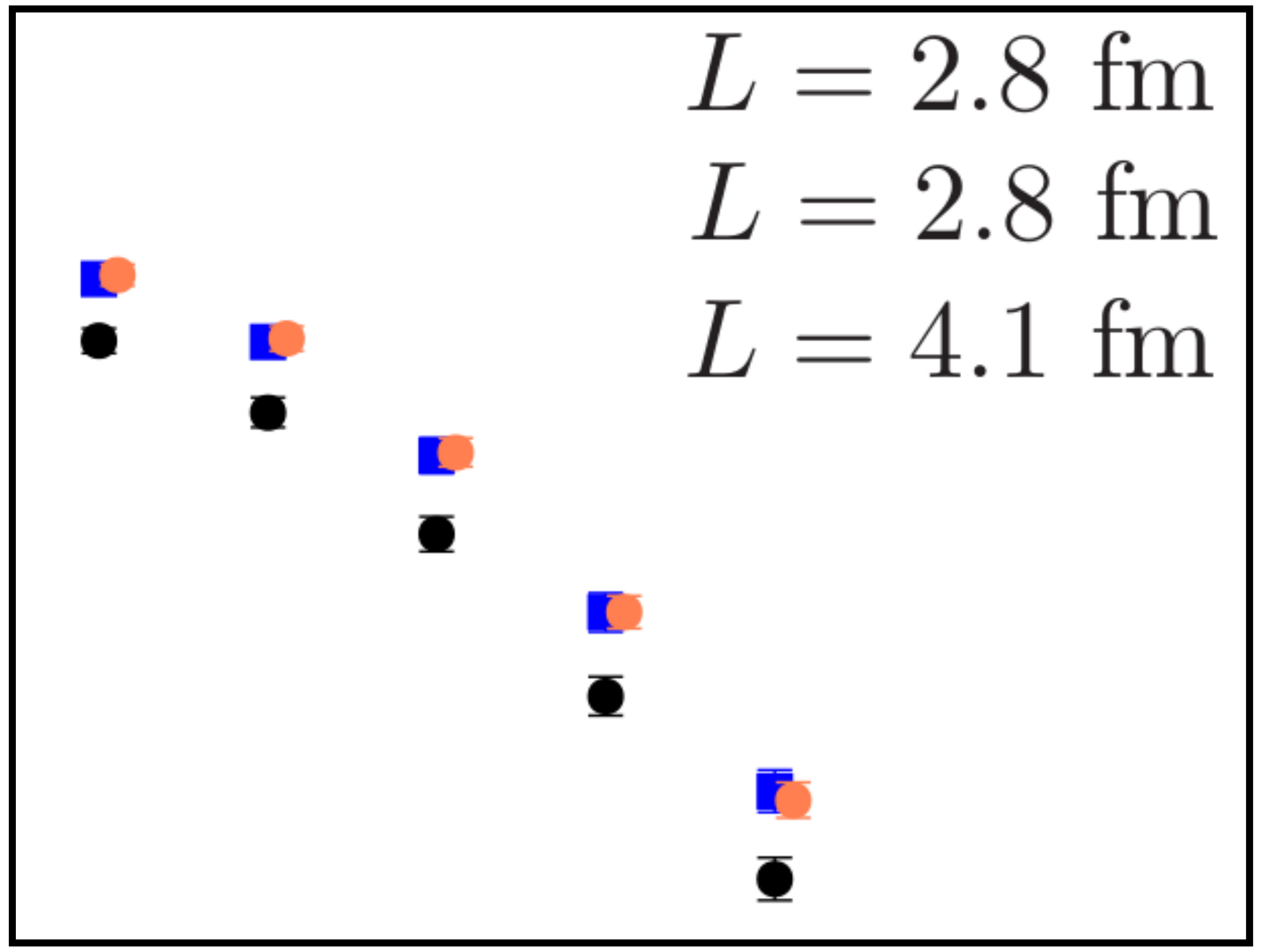
Finite-volume effects

Mainz method (aka MLL): *[Meyer 2011, Francis et al. 2013, Della Morte et al. 2017; Lellouch & Lüscher 2001]*

$$G(t,L) \overset{t\to\infty}{=} \sum_n |A_n|^2\, e^{-\omega_n t} \qquad G(t,\infty) = \int_0^\infty d\omega\, \omega^2 \rho(\omega^2)\, e^{-\omega|t|}$$

Both $|A_n|$ and $\rho(\omega^2)$ can be related to the pion form factor $F_\pi(\omega)$ $\Rightarrow$ $G(t,\infty) - G(t,L)$





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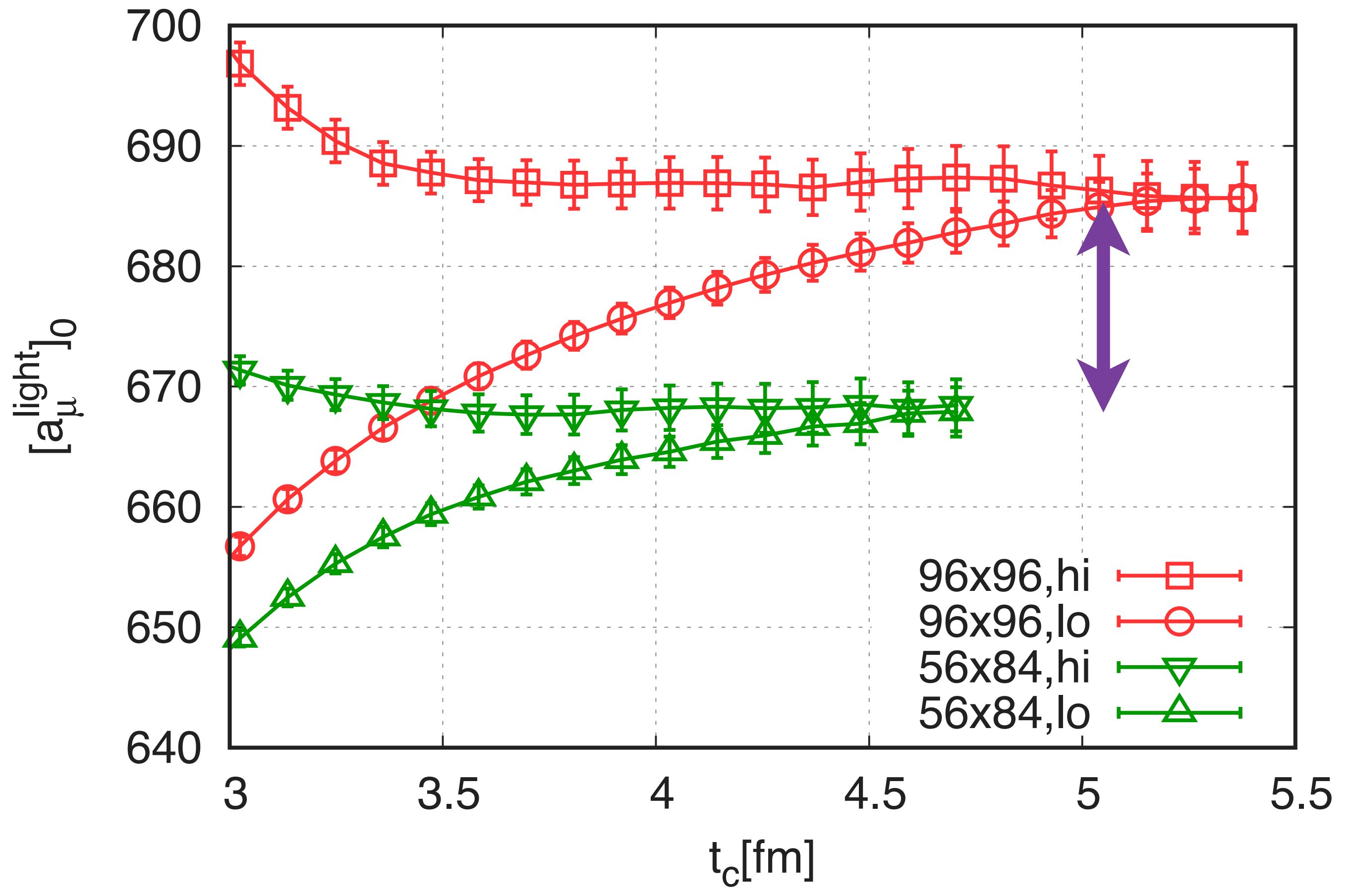
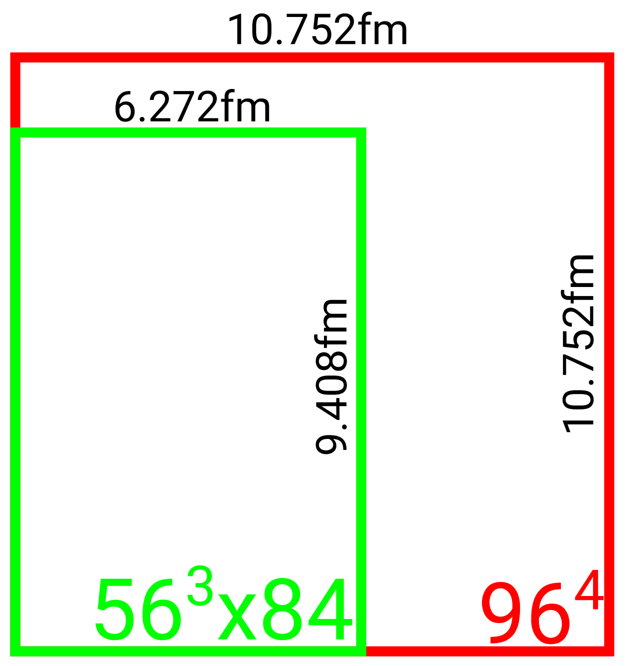
Other methods:

- Chiral Perturbation Theory *[Aubin et al. 2015,...]*
- Expansion in pion winding number *[Hansen & Patella]*

Correction	Comment
17.8	Gounaris-Sakurai model for $F_\pi(\omega)$
15.7	ChPT at NNLO
16.3	Expansion in pion winding number
18.1(2.4)	Direct lattice calculation

Direct lattice calculation by BMWc:





Scale setting

$$a_\mu^{\rm hvp,\,LO} \equiv a_\mu^{\rm hvp,\,LO}(M_\mu;\, M_u,\, M_d,\, M_s, \ldots) = \left(\frac{\alpha}{\pi}\right)^2 \int_0^\infty dt\, \tilde{K}(t)\, G(t), \quad M_\mu = m_\mu/\Lambda,\ M_u = m_u/\Lambda, \ldots$$

Λ : sets the lattice scale, e.g. $\Lambda = m_\Omega$

Kernel function $\tilde{K}(t)$ depends on dimensionless combination $(tm_\mu)^2$

[Della Morte et al 2017, Meyer & HW 2018]

Uncertainty in $a_\mu^{\rm hvp,\,LO}$ arising from scale uncertainty $\Delta\Lambda$:

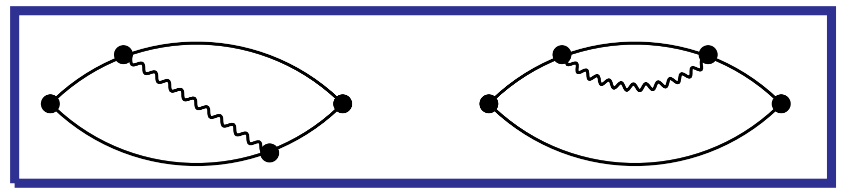
$$\frac{\Delta a_\mu^{\rm hvp,\,LO}}{a_\mu^{\rm hvp,\,LO}} = \left| \underbrace{\frac{M_\mu}{a_\mu^{\rm hvp,\,LO}} \frac{\partial a_\mu^{\rm hvp,\,LO}}{\partial M_\mu}}_{1.8} + \underbrace{\frac{M_\pi}{a_\mu^{\rm hvp,\,LO}} \frac{\partial a_\mu^{\rm hvp,\,LO}}{\partial M_\pi}}_{-0.18(6)} + \ldots \right| \frac{\Delta\Lambda}{\Lambda}$$

Requires precision calculation of the lattice scale!

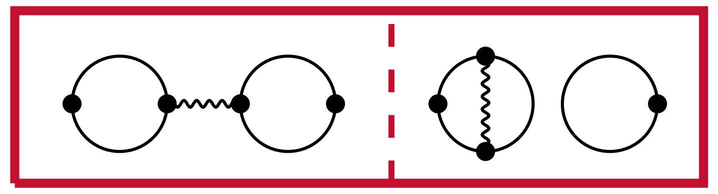
Must compute “bread-and-butter” quantities, such as hadron masses with $\mathrm{O}(0.1\%)$ precision

Scale setting also affects definition of the physical point via “hadronic renormalisation scheme”

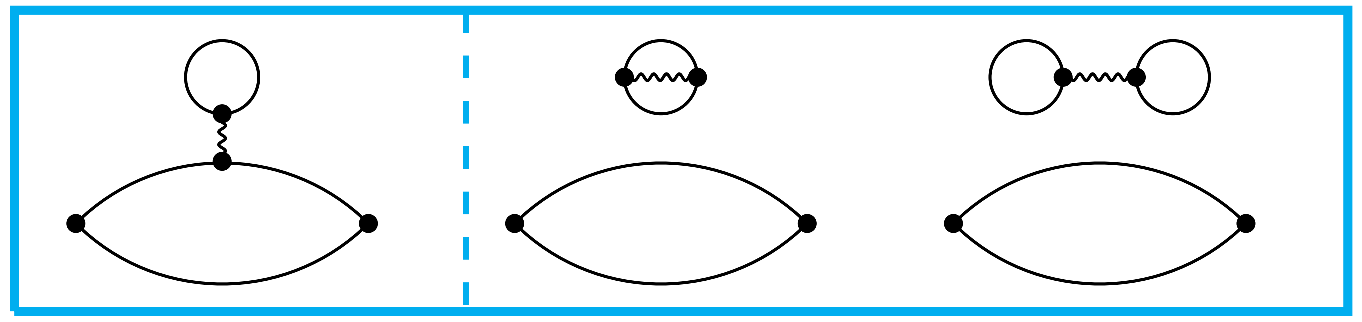
Isospin Breaking



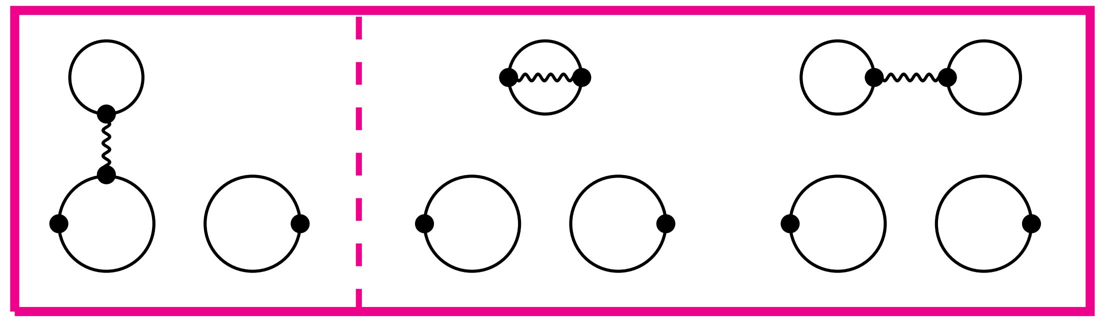
BMW −1.27(40)(33)
RBC/UKQCD 5.9(5.7)(1.7)
ETM 1.1(1.0)



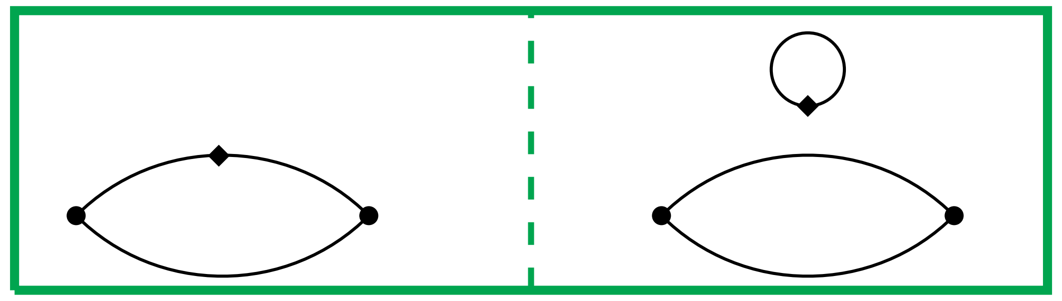
−0.55(15)(11) BMW
−6.9(2.1)(2.0) RBC/UKQCD



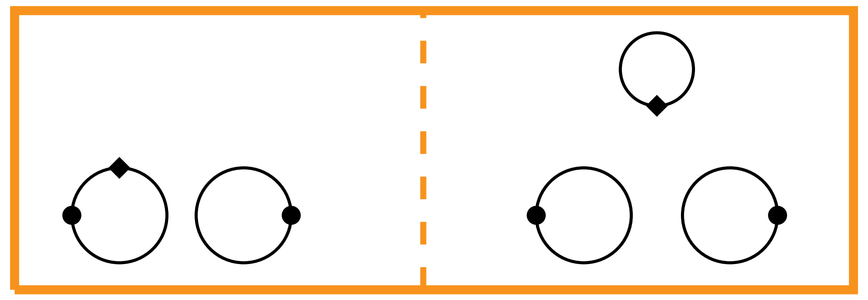
−0.0095(86)(99) 0.42(20)(19) BMW



0.011(24)(14) −0.047(33)(23) BMW



6.59(63)(53) BMW
10.6(4.3)(6.8) RBC/UKQCD
6.0(2.3) ETM
7.7(3.7) 9.0(2.3) FHM
9.0(0.8)(1.2) LM

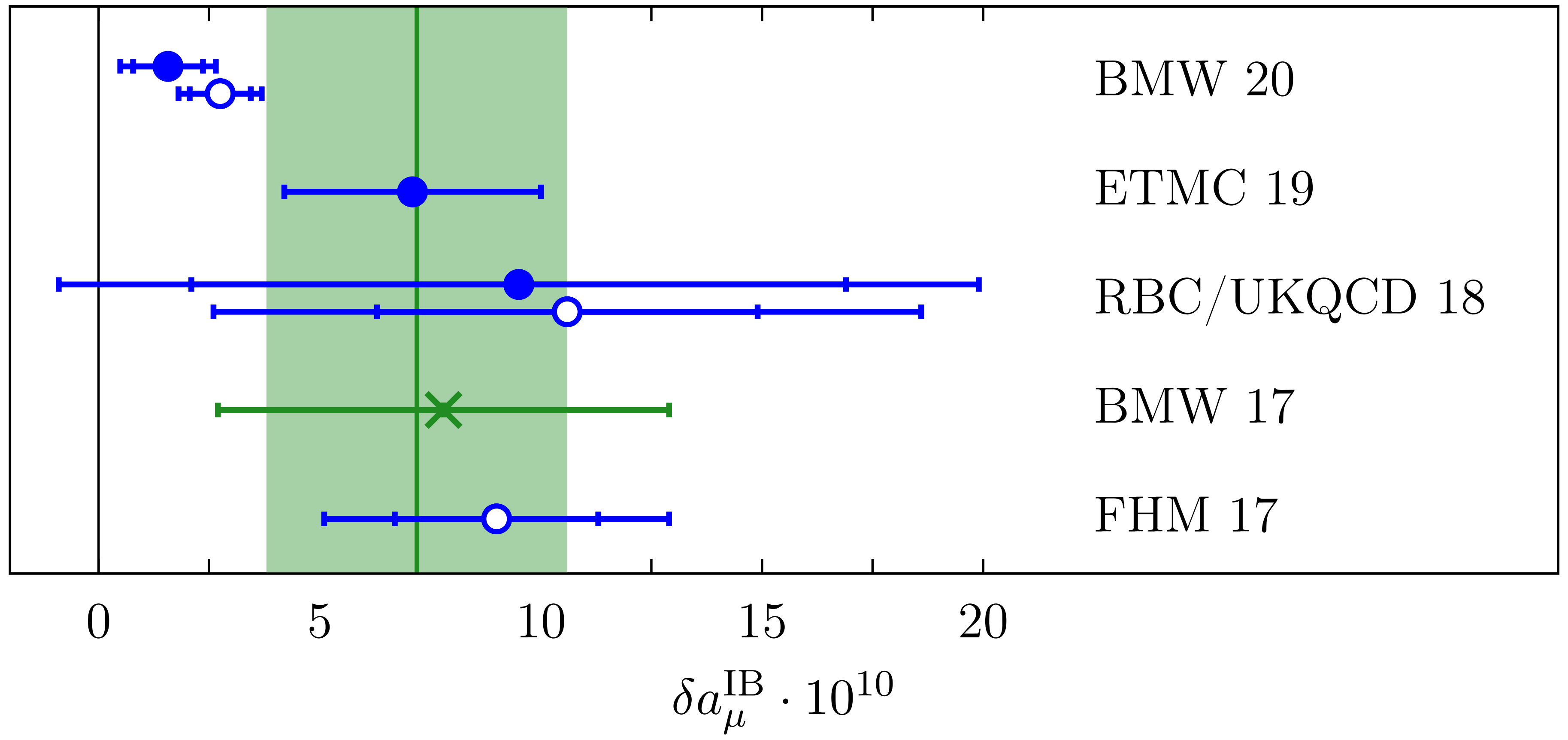


−4.63(54)(69) BMW

BMW [arXiv:2002.12347]
RBC/UKQCD [Phys.Rev.Lett. 121 (2018) 2, 022003]
ETM [Phys. Rev. D 99, 114502 (2019)]
FHM [Phys.Rev.Lett. 120 (2018) 15, 152001]
LM [Phys.Rev.D 101 (2020) 074515]

(Compilation by Vera Gülpers, Lattice-HVP Workshop Nov 2020)

Collection of published results:



- Small overall value result of cancellations
- Large statistical uncertainties:

$$a_\mu^{\mathrm{IB}} \lesssim 1\,\%\,, \quad \delta a_\mu^{\mathrm{IB}} \lesssim 100\,\%$$

- More precise calculations required

Window observables

Restric integration over Euclidean time to sub-intervals

→ reduce/enhance sensitivity to systematic effects

$$a_\mu^{\rm hvp,\,win} = \left(\frac{\alpha}{\pi}\right)^2 \int_0^\infty dt\, \tilde{K}(t)\, G(t)\, W(t;\, t_0,\, t_1)$$

Short distance: $W^{\rm SD}(t;\, t_0) = 1 - \Theta(t,\, t_0, \Delta)$

Intermediate distance: $W^{\rm ID}(t;\, t_0,\, t_1) = \Theta(t,\, t_0, \Delta) - \Theta(t,\, t_1, \Delta)$

Long distance: $W^{\rm LD}(t;\, t_1) = \Theta(t,\, t_1, \Delta)$

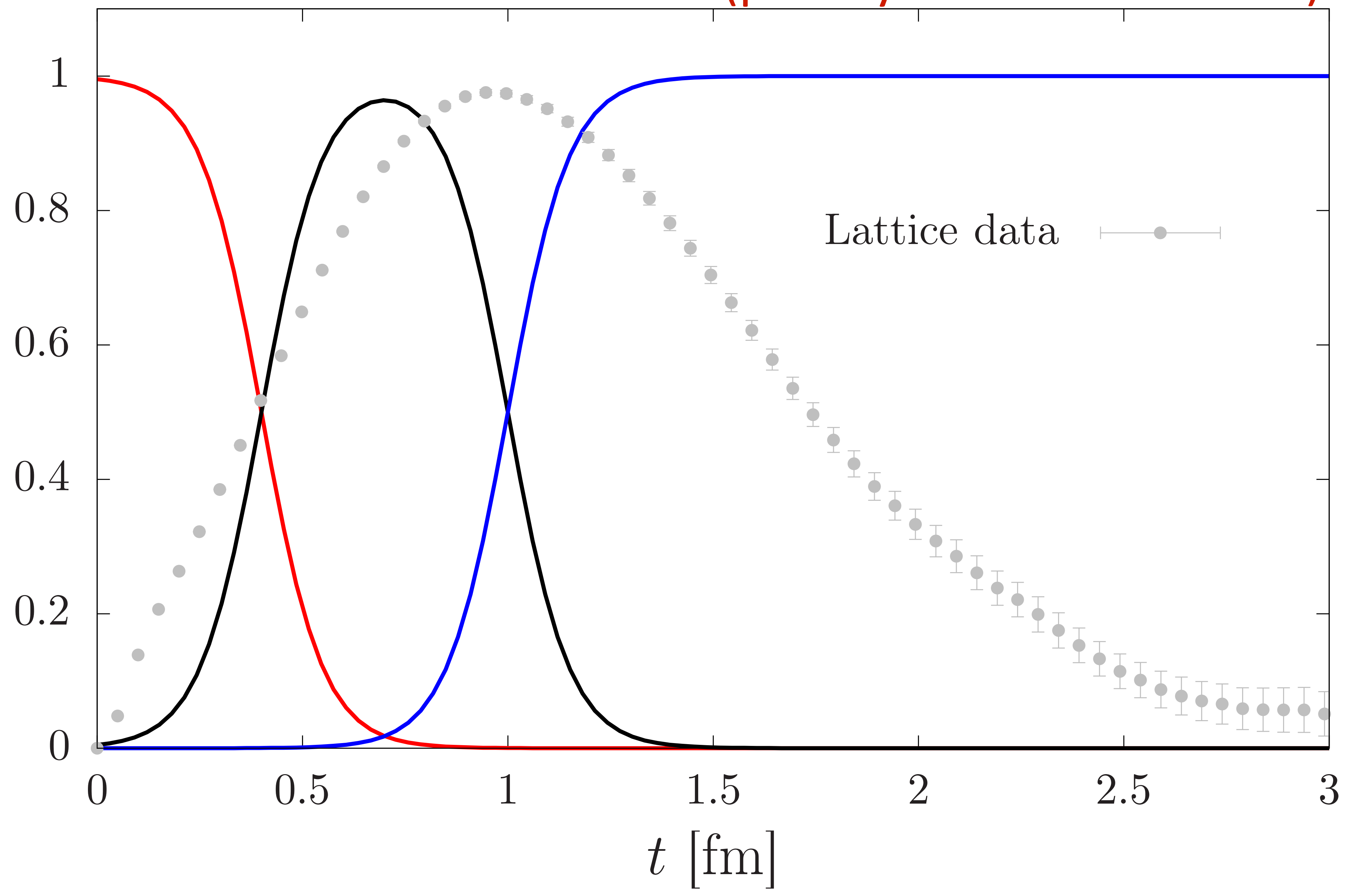
$$\Theta(t,\, t', \Delta) = \tfrac{1}{2}\left[1 + \tanh(t - t')/\Delta\right]$$

[RBC/UKQCD 2018]

"Standard" choice:

$$t_0 = 0.4\,\text{fm},\ t_1 = 1.0\,\text{fm},\ \Delta = 0.15\,\text{fm}$$

(plot by Antoine Gérardin)



Intermediate window:

- Finite-volume correction reduced from 3% to 0.25%
- Uncertainty dominated by statistics

⇒ Precision test of different lattice calculations

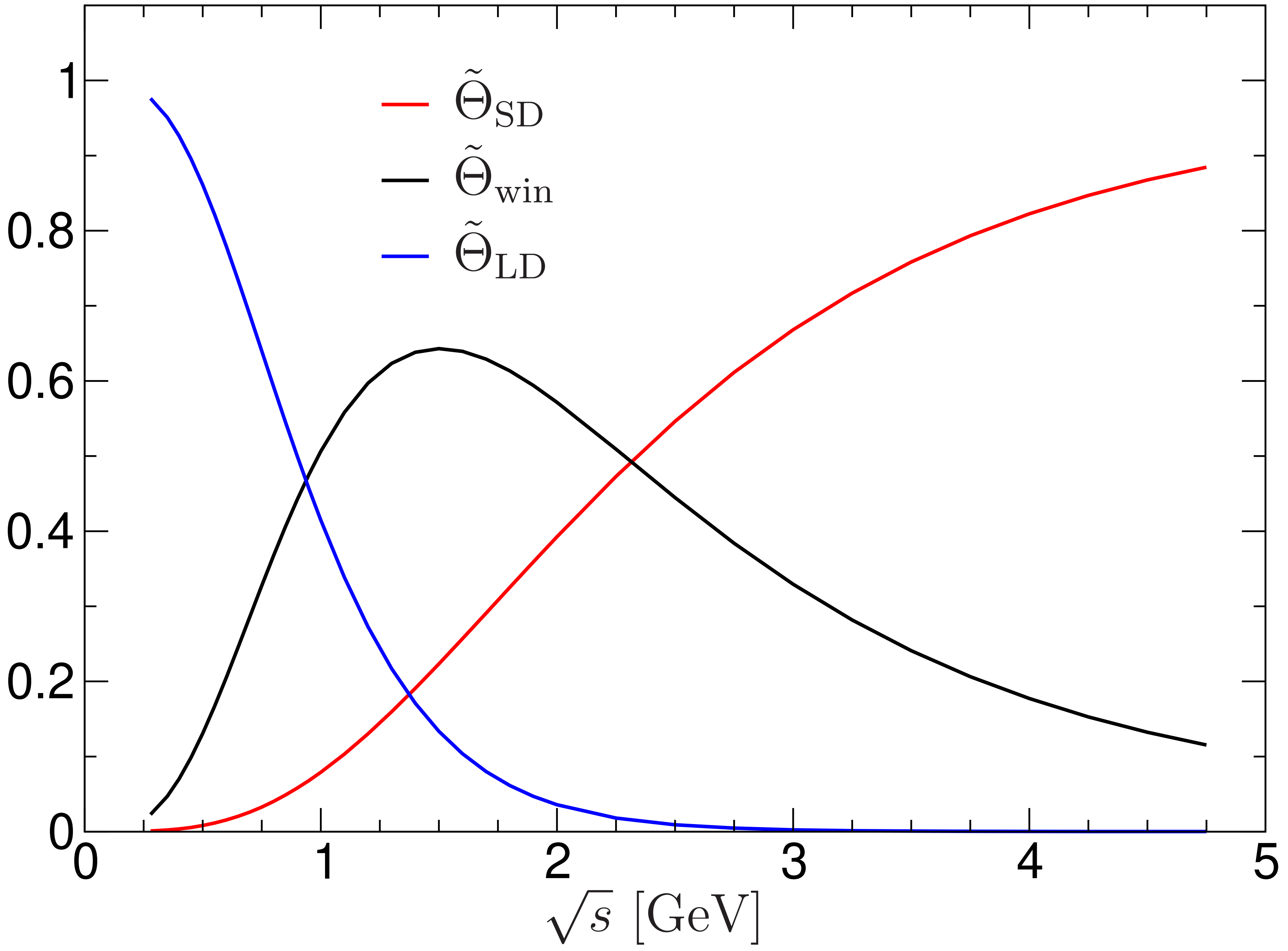
⇒ Comparison with corresponding R-ratio estimate

Window observables: Comparison with R-ratio

Starting point: $$G(t) = \frac{1}{12\pi^2}\int_{m_{\pi^0}^2}^{\infty} d(\sqrt{s})\,R(s)\,s\,\mathrm{e}^{-\sqrt{s}t}$$ [RBC/UKQCD 2018]

Insert $G(t)$ into expression for time-momentum representation:

$$a_\mu^{\mathrm{hvp,\,ID}} = \left(\frac{\alpha}{\pi}\right)^2 \int_{m_{\pi^0}^2}^{\infty} d(\sqrt{s})\,R(s)\,\frac{1}{12\pi^2}\,s\int_0^\infty dt\,\tilde{K}(t)\,W^{\mathrm{ID}}(t;\,t_0,\,t_1)\,\mathrm{e}^{\sqrt{s}t}$$



Intermediate window from R-ratio following procedure for WP estimate:

$$a_\mu^{\mathrm{hvp,\,ID}} \equiv a_\mu^{\mathrm{win}} = (229.4 \pm 1.4)\cdot 10^{-10}$$

Finer decomposition allows for more detailed studies of energy dependence

[Colangelo et al., arXiv:2205.12963]

Hadronic running of $\alpha_{\mathrm{e.m.}}$

Correlation between a_μ and the hadronic running of α :

$$\Delta\alpha_{\mathrm{had}}^{(5)}(M_Z^2) = \frac{\alpha\, M_Z^2}{3\pi}\, \mathrm{P}\int_{m_{\pi^0}^2}^{\infty} ds\, \frac{R_{\mathrm{had}}(s)}{s(M_Z^2 - s)}$$

→ Input quantity for global electroweak fit

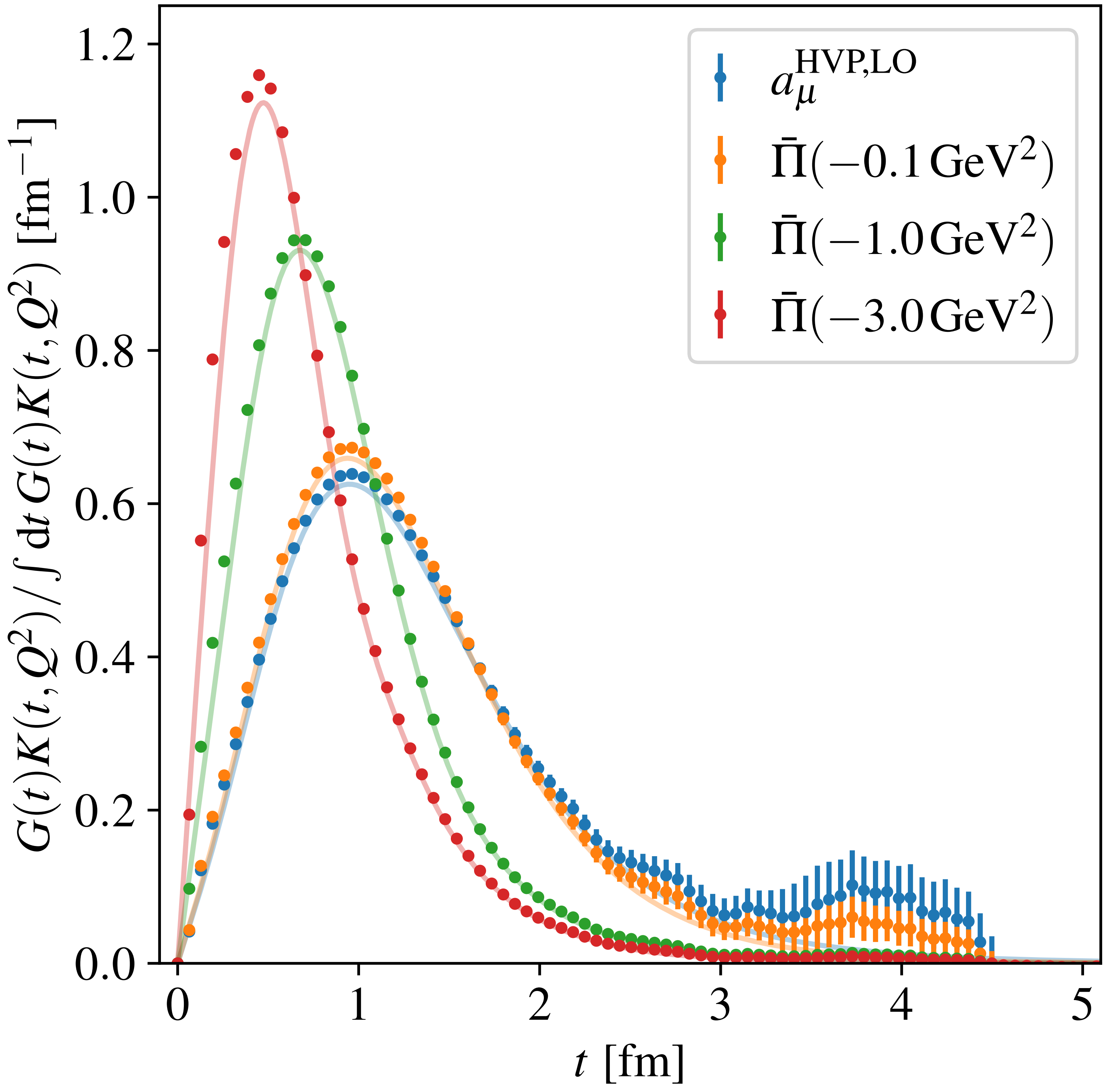
Larger estimate for $a_\mu^{\mathrm{hvp,\,LO}}$ implies an increase in $\Delta\alpha_{\mathrm{had}}^{(5)}(M_Z^2)$

High-energy regime of R-ratio weighted more strongly

Lattice calculation of $\Delta\alpha_{\mathrm{had}}(-Q^2)$ based on the same correlator $G(t)$ with a different kernel function:

$$\Delta\alpha_{\mathrm{had}}(-Q^2) = \frac{\alpha}{\pi}\frac{1}{Q^2}\int_0^{\infty} dt\, G(t)\left[Q^2t^2 - 4\sin^2\left(\tfrac{1}{2}Q^2t^2\right)\right]$$

Profile of the integrand of $\Delta\alpha_{\mathrm{had}}(-1\,\mathrm{GeV}^2)$ resembles that of $a_\mu^{\mathrm{hvp,\,ID}}$



Discretisations of the quark action

Rooted staggered quarks:

- remnant fermion doublers — "tastes"
- correct analytically for taste-induced lattice artefacts
- low computational cost
- used by BMWc, FHM, Lehner & Meyer, Aubin et al., χQCD

Wilson quarks:

- no doublers; chiral symmetry broken explicitly
- leading lattice artefacts of $\mathrm{O}(a^2)$ after Symanzik improvement, twisted-mass formalism
- moderate computational cost
- used by ETMC, Mainz/CLS, PACS

Domain wall /overlap quarks (Ginsparg-Wilson quarks):

- no doublers; chiral symmetry breaking exponentially small
- live in five dimensions (dwf); evaluate sign function of "conventional" discretisation (ovlp)
- leading lattice artefacts of $\mathrm{O}(a^2)$
- high computational cost
- used by RBC/UKQCD, BMWc (Ovlp valence), χQCD (DWF sea, Ovlp valence)

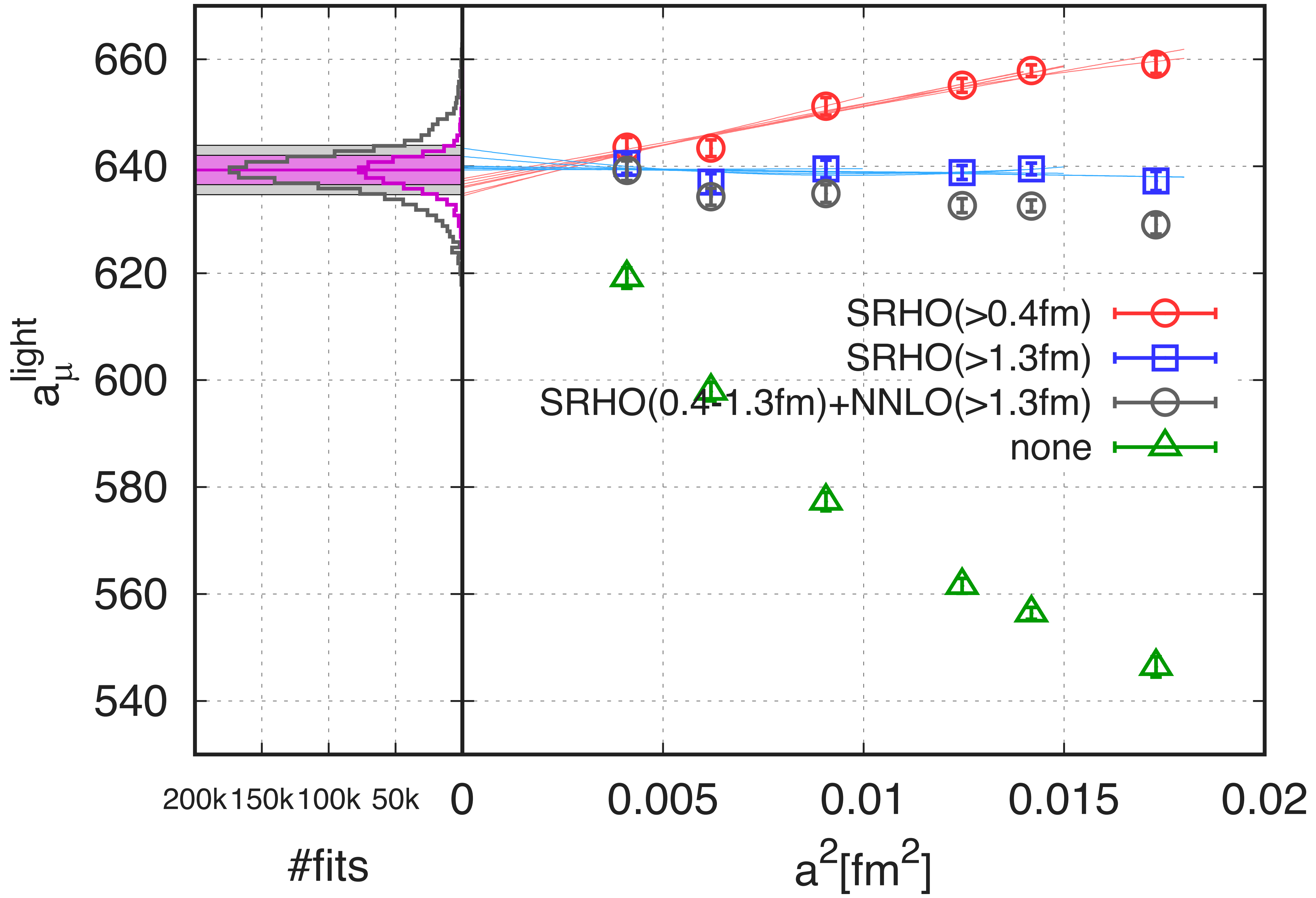
Staggered fermions

Budapest-Marseille-Wuppertal Collaboration

[Borsányi et al., Nature 593 (2021) 7857, arXiv:2002.12347v3]

- $N_f = 2 + 1 + 1$ flavours; one-link staggered fermion action with four steps of stout smearing; six lattice spacings: $a = 0.132 - 0.064\,\mathrm{fm}$, physical pion mass
- Correct for taste-breaking effects using the SRHO model, combined with SChPT
- Comprehensive study of finite-volume and isospin-breaking corrections
- Final result selected from distribution of different fits
- Results dominated by systematic error associated with continuum extrapolation

$$a_\mu^{\mathrm{hvp,LO}} = (707.5 \pm 2.3 \pm 5.0) \cdot 10^{-10}$$



Staggered fermions

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- Final result selected from distribution of different fits
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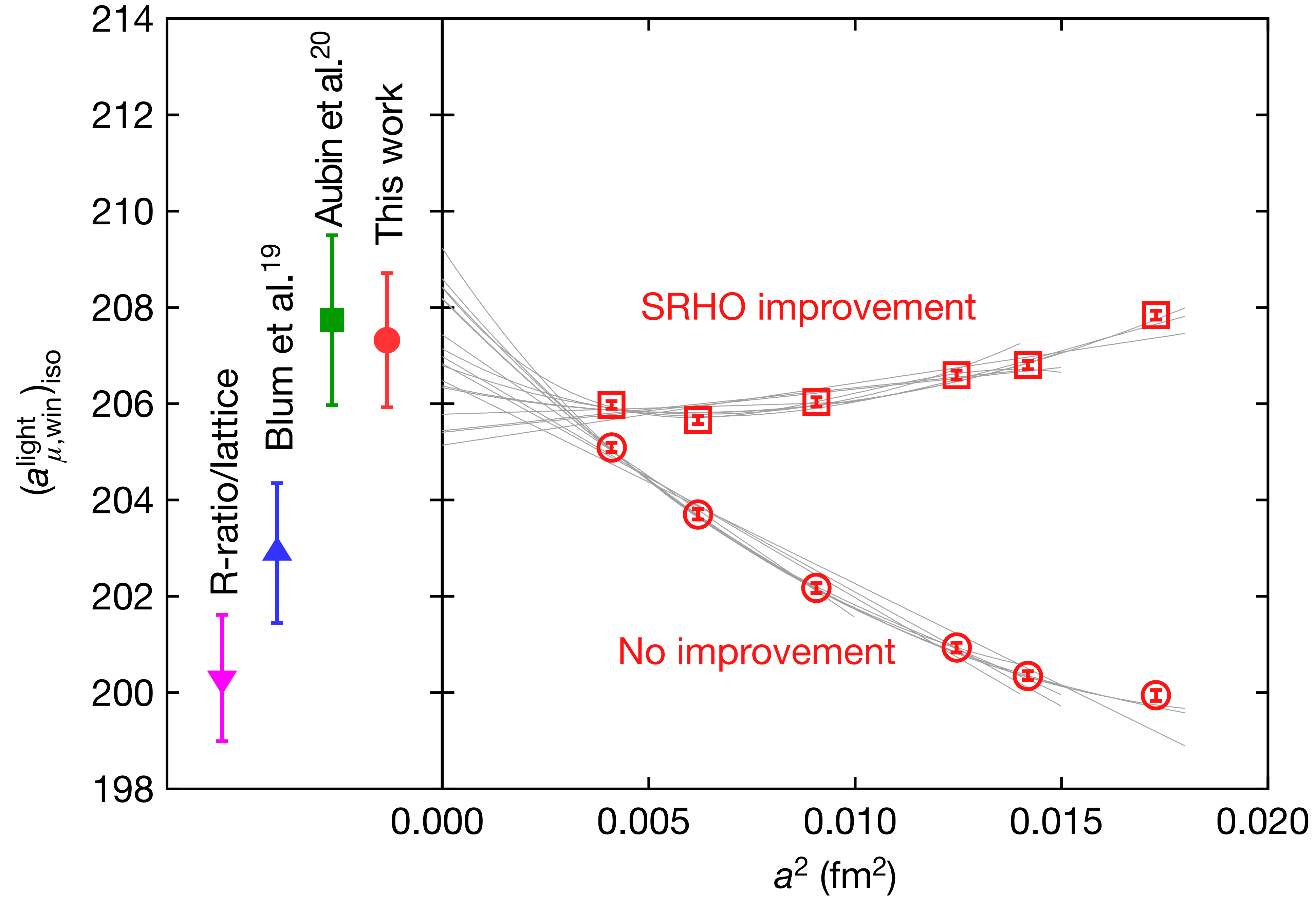
$$a_\mu^{\text{hvp, LO}} = (707.5 \pm 2.3 \pm 5.0) \cdot 10^{-10}$$

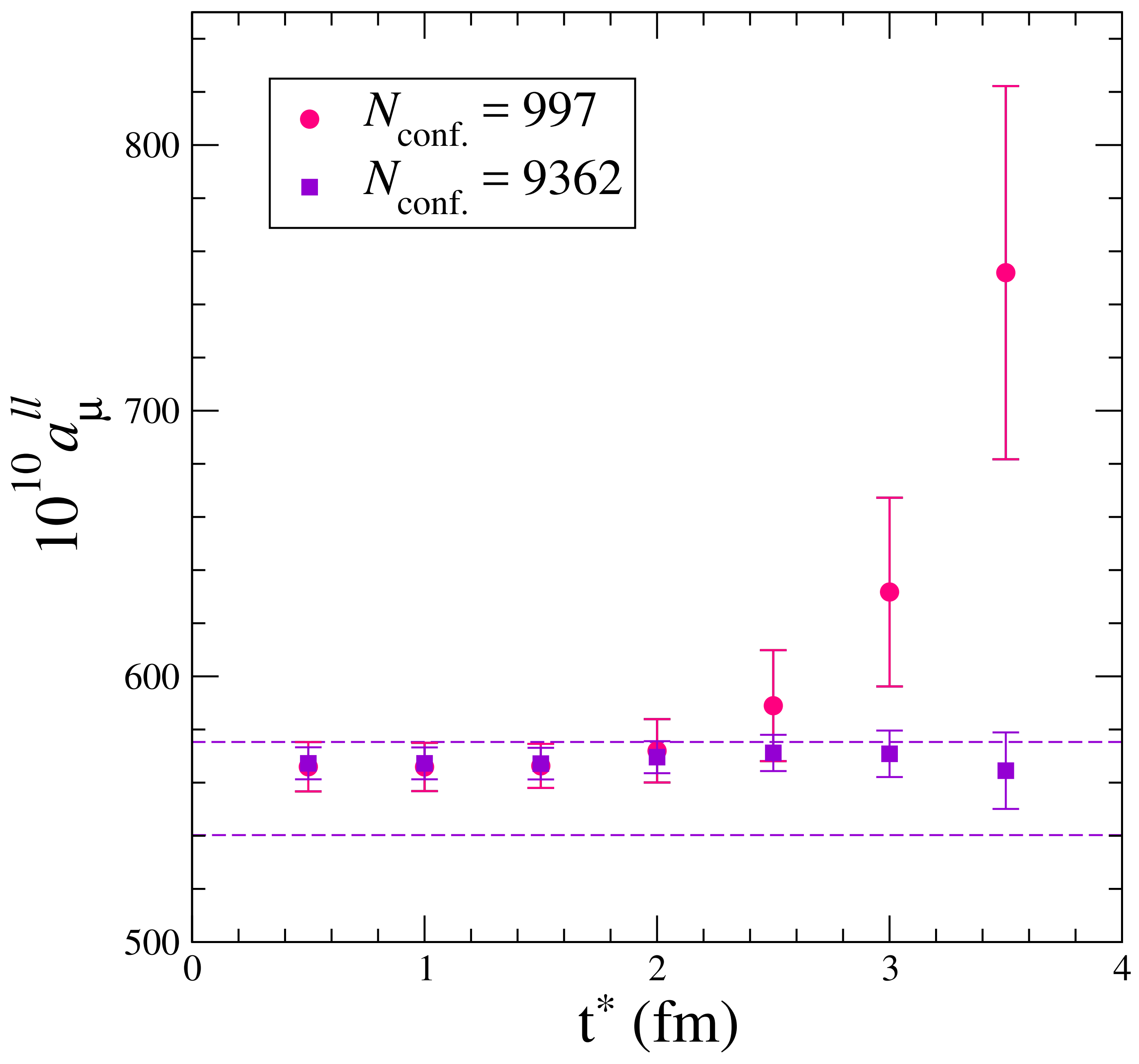
$$a_\mu^{\text{win,ud}} = (207.3 \pm 0.4 \pm 1.3) \cdot 10^{-10}$$

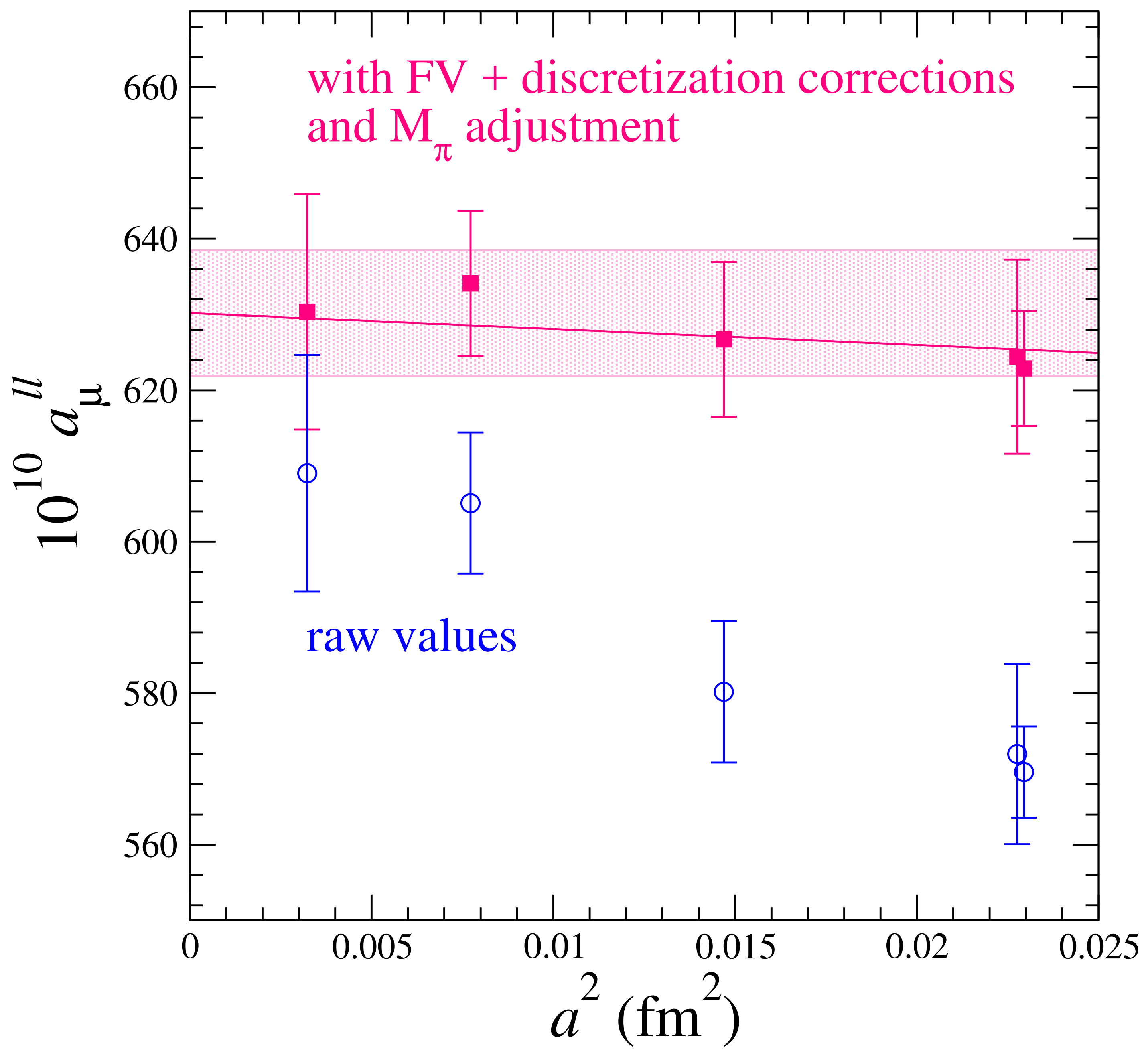


Fermilab-HPQCD-MILC Collaboration

[Davies et al., Phys.Rev.D 101 (2020) 3, 034512]

- $N_f = 2+1+1$ flavours; highly-improved staggered quark (HISQ) action; four lattice spacings: $a = 0.15 - 0.06\,\mathrm{fm}$, physical pion mass
- Correct for taste-breaking and finite-volume effects using the SRHO model
- Simulations with $N_f = 1+1+1+1$ to determine isospin-breaking corrections
- Multi-state fits to isolate long-distance behaviour of $G(t)$; comparison with bounding method

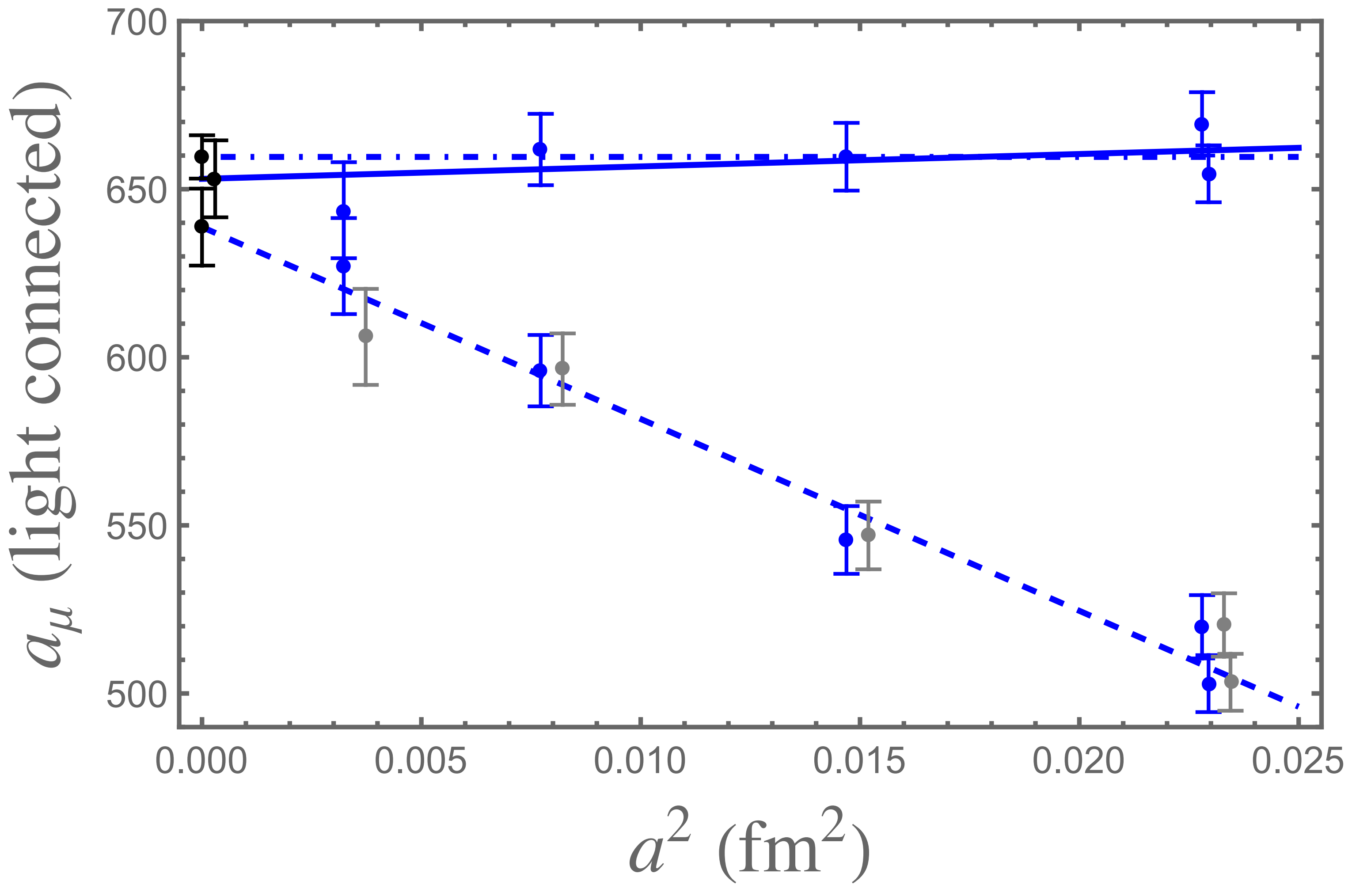


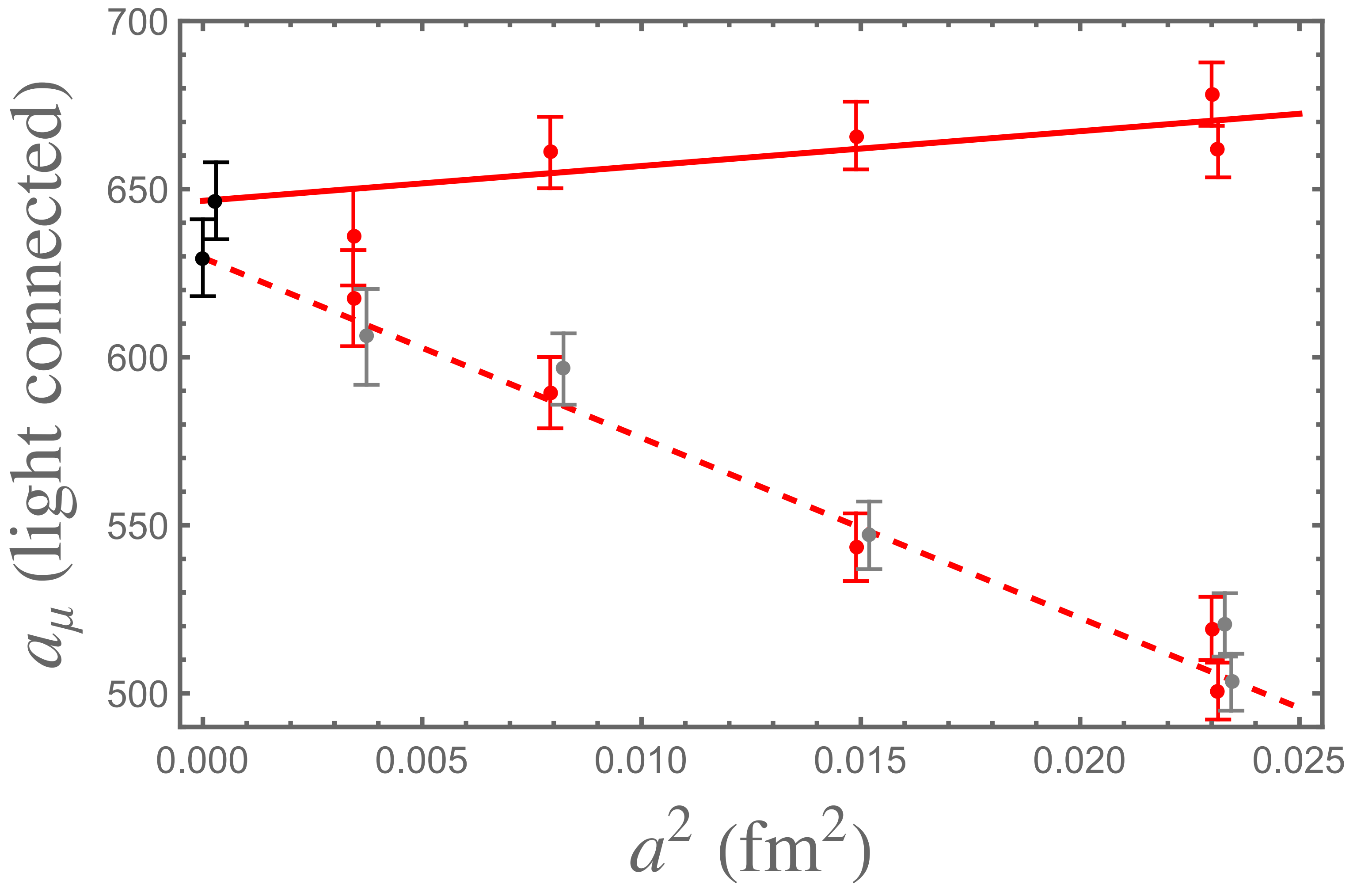


$$a_\mu^{\mathrm{hvp,LO}} = (699 \pm 15) \cdot 10^{-10}$$

Aubin, Blum, Golterman & Peris *[Phys.Rev.D 101 (2020) 014503; arXiv:2204.12256]*

- $N_f = 2 + 1 + 1$ flavours; highly-improved staggered quark (HISQ) action; four lattice spacings: $a = 0.15 - 0.06\,\mathrm{fm}$, physical pion mass
- Compute finite-volume and taste-breaking corrections using SChPT at NNLO; compare with SRHO
- Standard intermediate window: SChPT / SRHO ceases to be a systematically improvable formalism
- Extrapolations of corrected / uncorrected data:





$$a_\mu^{\mathrm{hvp,\,ud}} = (646 \pm 14) \cdot 10^{-10}$$

$$a_\mu^{\mathrm{hvp,\,LO}} = (701 \pm 14) \cdot 10^{-10}*$$

*Combination with results from other calculations

Aubin, Blum, Golterman & Peris [Phys.Rev.D 101 (2020) 014503; arXiv:2204.12256]

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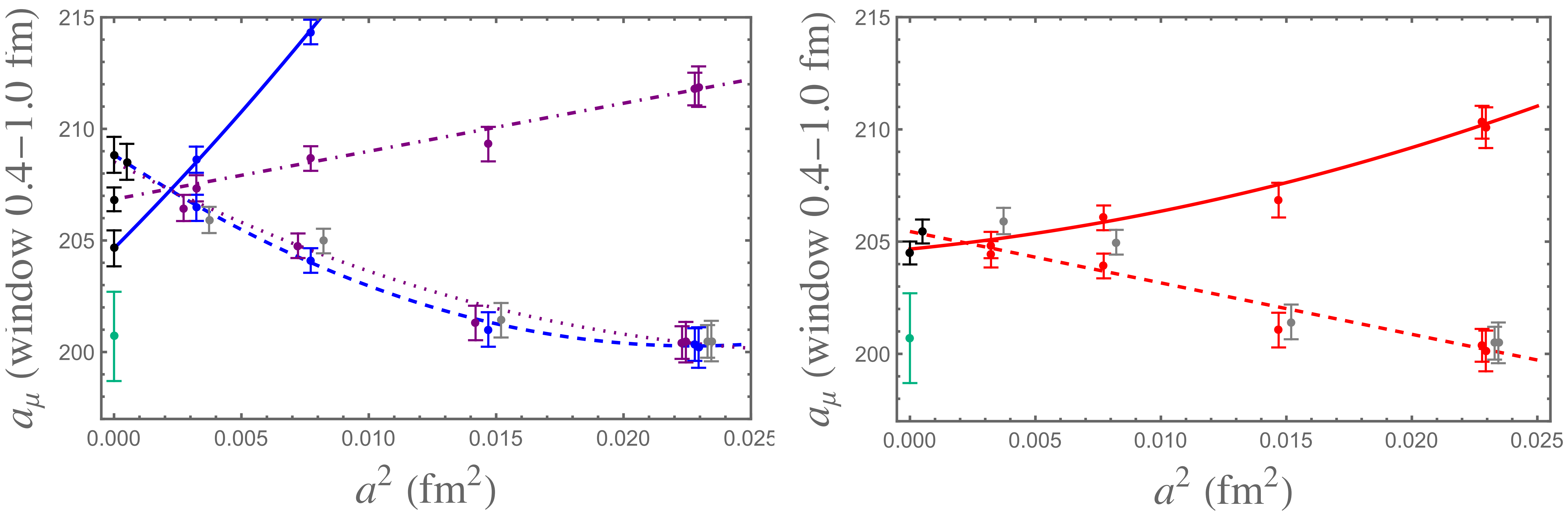


$$a_\mu^{\mathrm{hvp,\,ud}} = (646 \pm 14)\cdot 10^{-10}$$

$$a_\mu^{\mathrm{hvp,\,LO}} = (701 \pm 14)\cdot 10^{-10}\,*$$

$$a_\mu^{\mathrm{win,\,ud}} = (206.8 \pm 2.2)\cdot 10^{-10}$$

"Taste-breaking effects appear to be the largest effect hindering a straightforward extrapolation to the continuum limit" → include smaller lattice spacings

*Combination with results from other calculations

Domain wall fermions

RBC/UKQCD Collaboration

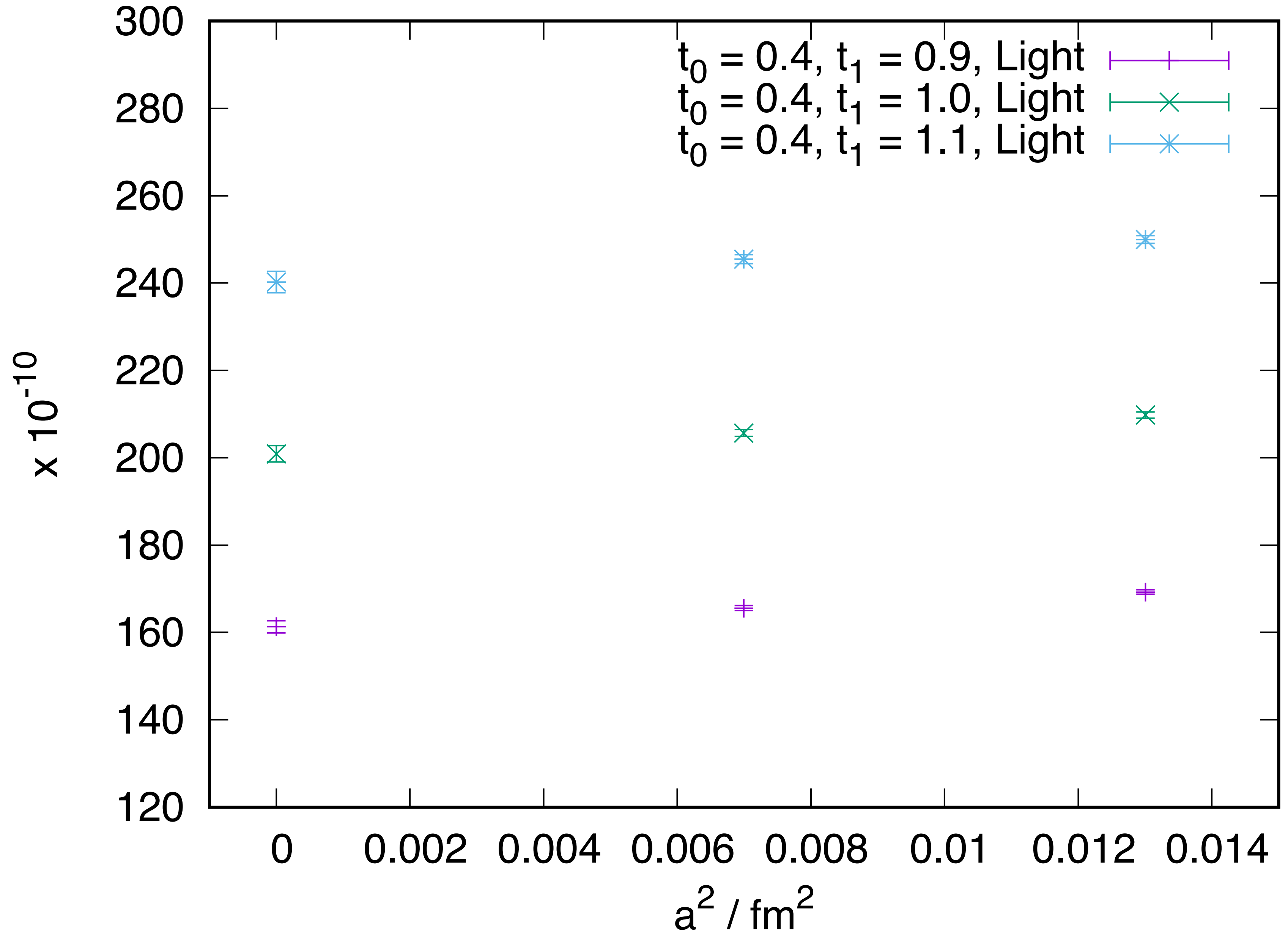
[Blum et al., Phys. Rev. Lett. 121 (2018) 022003]

- $N_f = 2+1$ flavours; domain wall action; two ensembles at $a = 0.114,\ 0.084\,\mathrm{fm}$, physical pion mass
- Finite-volume corrections computed in ChPT
- Strong and QED isospin-breaking corrections computed, excluding disconnected diagrams
- Definition of the "window observables"
- Approach to continuum limit consistent with a^2-behaviour plus small a^4-term estimated by power-counting arguments

$$a_\mu^{\mathrm{hvp,\,ud}} = (649.7 \pm 14.2 \pm 4.9)\cdot 10^{-10}$$

$$a_\mu^{\mathrm{hvp,\,LO}} = (715.4 \pm 16.3 \pm 9.2)\cdot 10^{-10}$$

$$a_\mu^{\mathrm{win,\,ud}} = (202.9 \pm 1.4 \pm 0.4)\cdot 10^{-10}$$



- Third lattice spacing at $a \approx 0.07\,\mathrm{fm}$ in preparation

Mixed action calculations

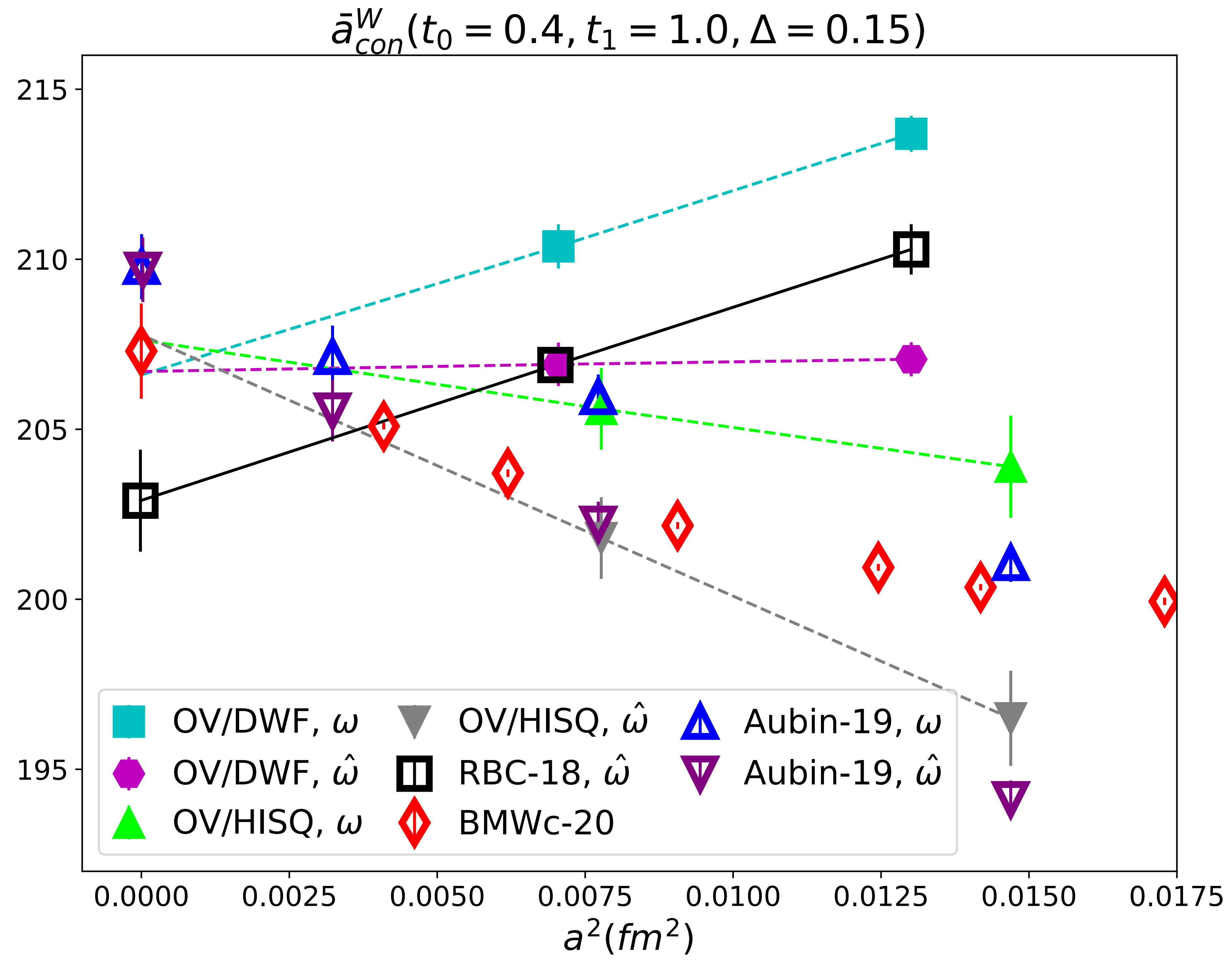
χQCD Collaboration

[Wang et al., arXiv:2204.01280]

- Overlap valence quarks on HISQ $(N_f = 2 + 1 + 1)$ and Domain-wall $(N_f = 2 + 1)$ gauge ensembles
- Lattice spacings: $a = 0.114,\ 0.084\,\mathrm{fm}$ (DWFs) and $a = 0.12,\ 0.09\,\mathrm{fm}$ (HISQ) — physical pion mass
- Scaling test at $m_\pi = 320\,\mathrm{MeV}$ using four HISQ ensembles with $a = 0.12 - 0.04\,\mathrm{fm}$
- Focus on intermediate window observable
- Linear extrapolations in a^2 produce consistent results in both mixed-action setups

$$a_\mu^{\mathrm{win,\,ud}} = (206.7 \pm 1.5) \cdot 10^{-10} \qquad \text{(Ovlp/DWF)}$$

$$a_\mu^{\mathrm{win,\,ud}} = (207.7 \pm 3.1) \cdot 10^{-10} \qquad \text{(Ovlp/HISQ)}$$



Wilson fermions

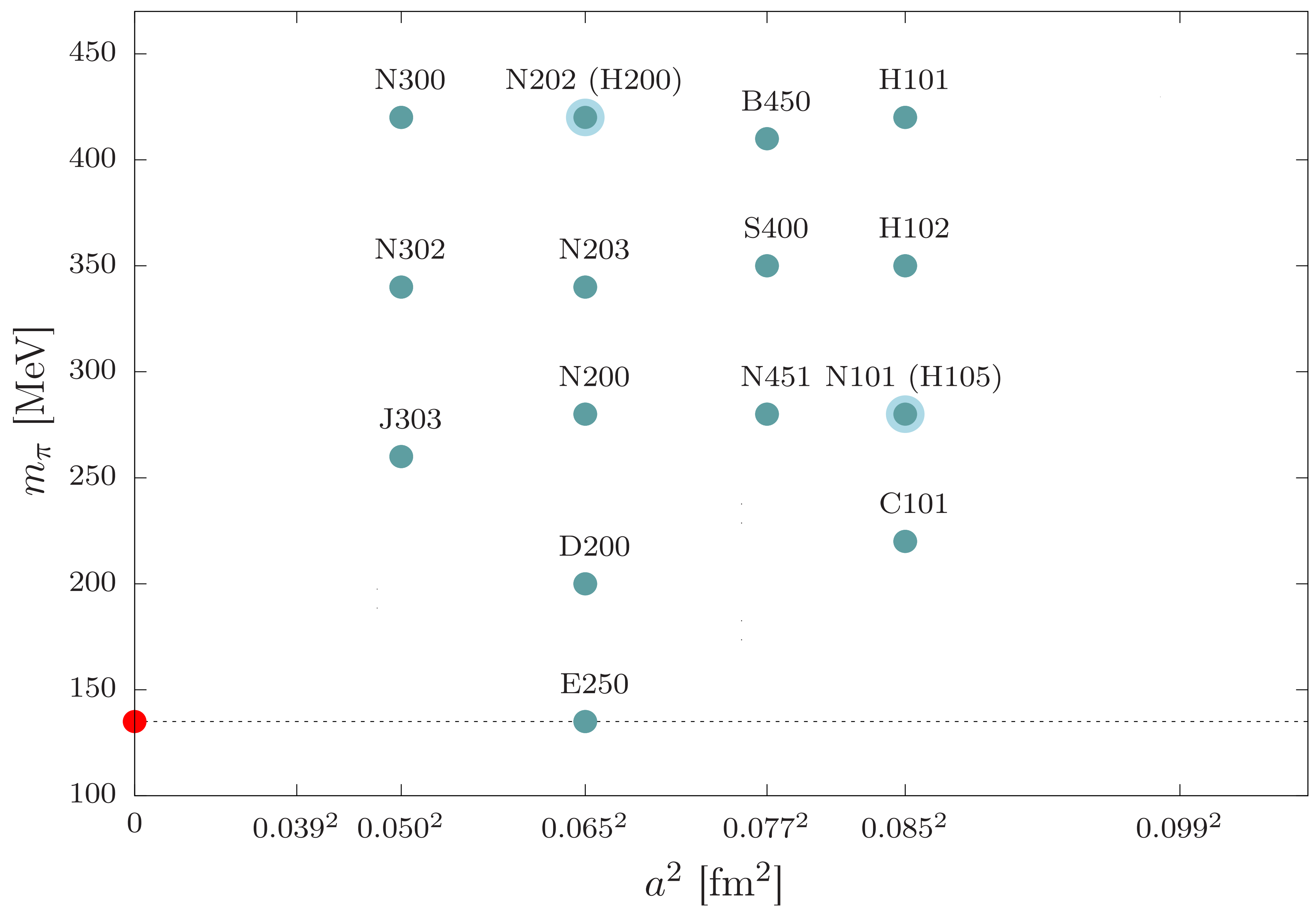
Mainz/CLS

[Gérardin et al., Phys. Rev. D 100 (2019) 014510, Cè et al., arXiv:2206.06582]

- $N_f = 2 + 1$ flavours of $\mathrm{O}(a)$ improved Wilson fermions
- Four lattice spacings: $a = 0.085 - 0.050\,\mathrm{fm}$; pion masses $m_\pi = 130 - 420\,\mathrm{MeV}$
- Two discretisations of the vector current: local and conserved
- Simultaneous chiral and continuum extrapolation

$$a_\mu^{\mathrm{hvp,\,LO}} = (720.0 \pm 12.6 \pm 9.9) \cdot 10^{-10}$$

(Isospin-breaking correction computed by ETMC added to error)



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- Simultaneous chiral and continuum extrapolation

$$a_\mu^{\mathrm{hvp,\,LO}} = (720.0 \pm 12.6 \pm 9.9) \cdot 10^{-10}$$

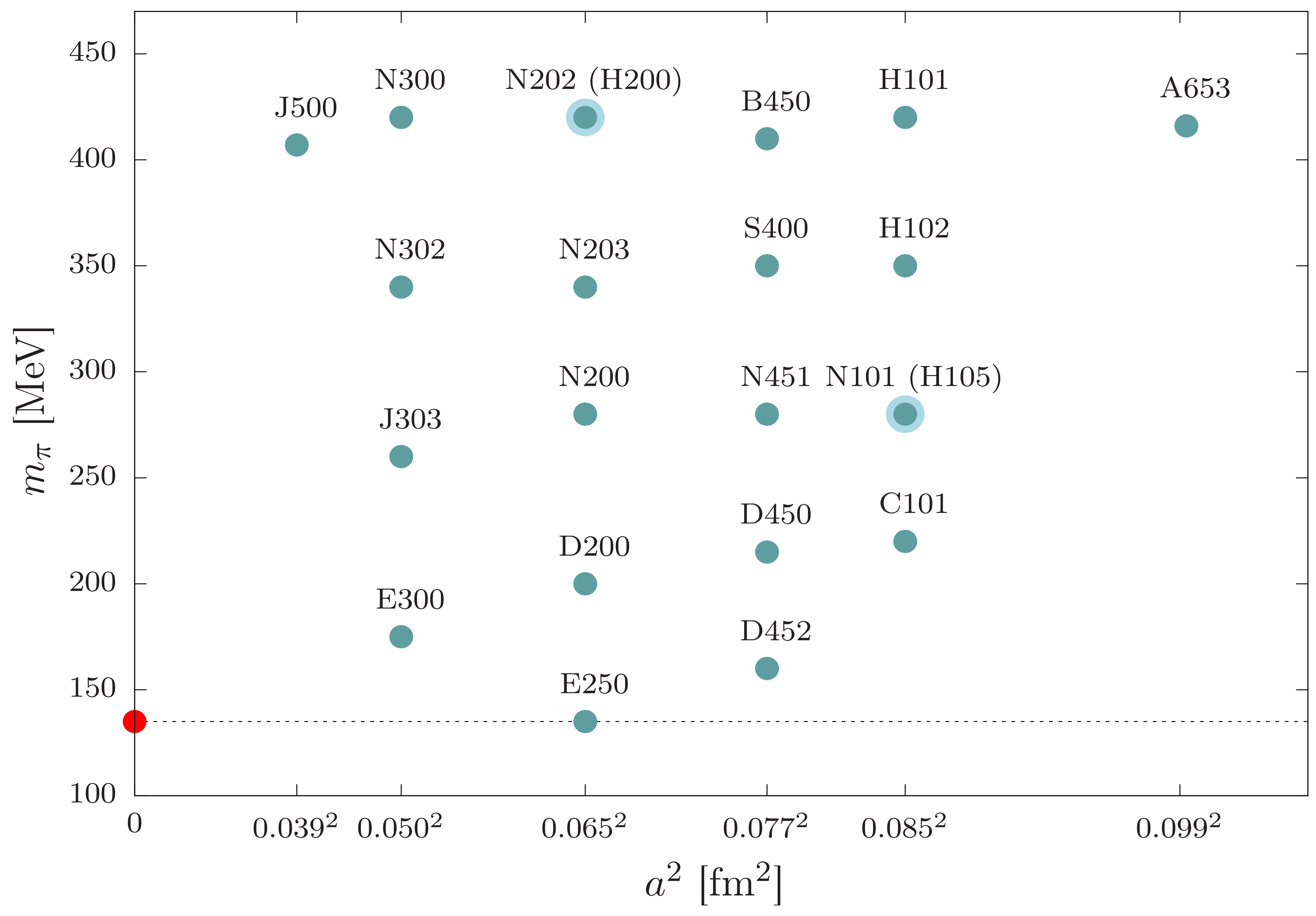
(Isospin-breaking correction computed by ETMC added to error)

- Extension to six lattice spacings: $a = 0.099 - 0.039\,\mathrm{fm}$; additional ensembles with $m_\pi \gtrsim m_\pi^{\mathrm{phys}}$

$$a_\mu^{\mathrm{win,\,ud}} = (207.00 \pm 0.83 \pm 1.20) \cdot 10^{-10}$$

$$a_\mu^{\mathrm{win}} = (237.30 \pm 0.79 \pm 1.22) \cdot 10^{-10}$$

(Isospin-breaking corrections computed and included)



Mainz/CLS: Scaling test

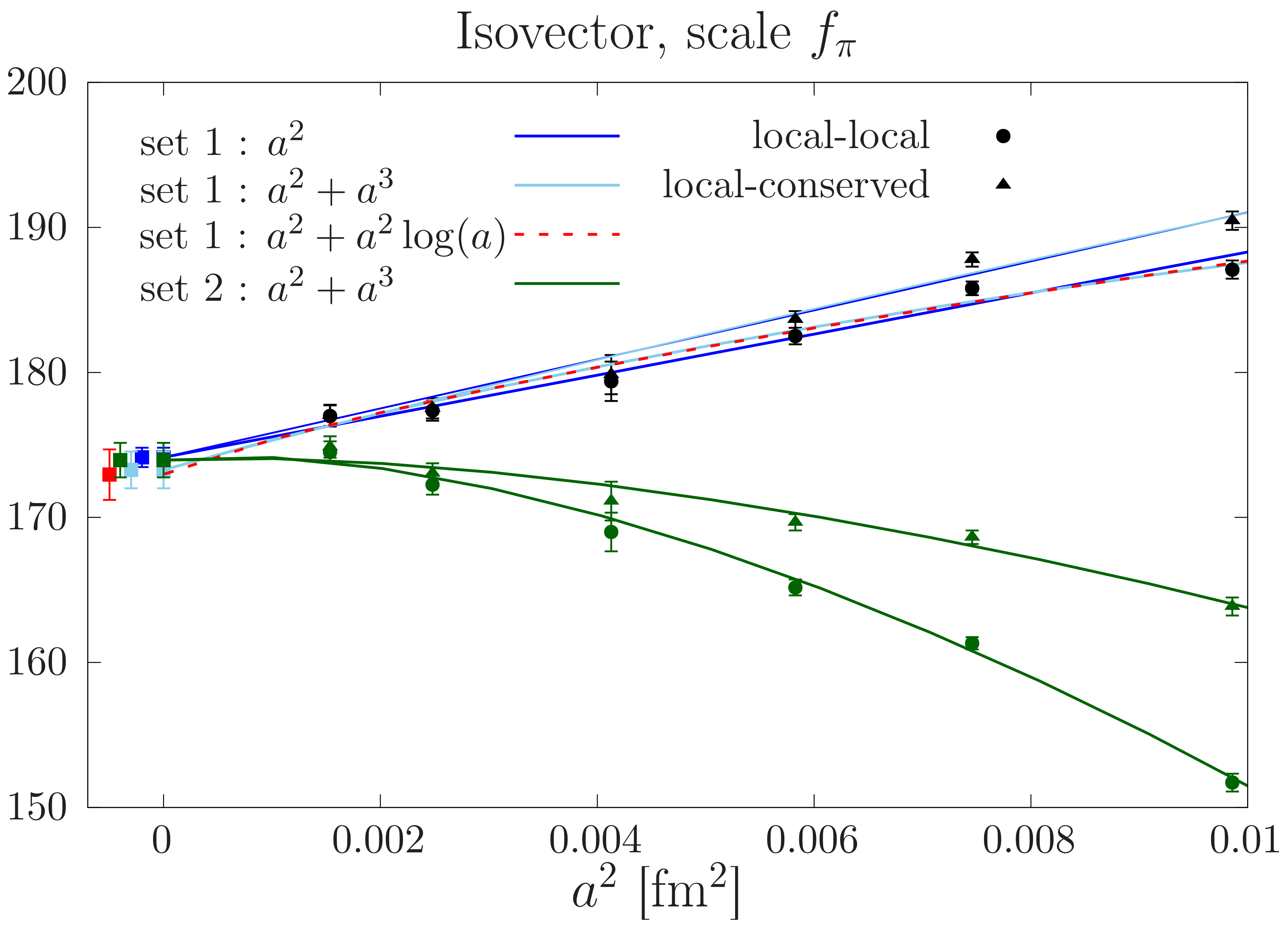
[Cè et al., arXiv:2206.06582]

- Two independent sets of improvement coefficients for local and conserved currents

$$J_\mu^{(\alpha)} = j_\mu^{(\alpha)} + a\, c_{\rm V}(g_0)\, \tilde{\partial}_\nu \Sigma_{\mu\nu}, \qquad \alpha = L,\, C$$

→ four different discretisations of the current-current correlator

- Scaling test at $m_\pi = 420\,\mathrm{MeV}$



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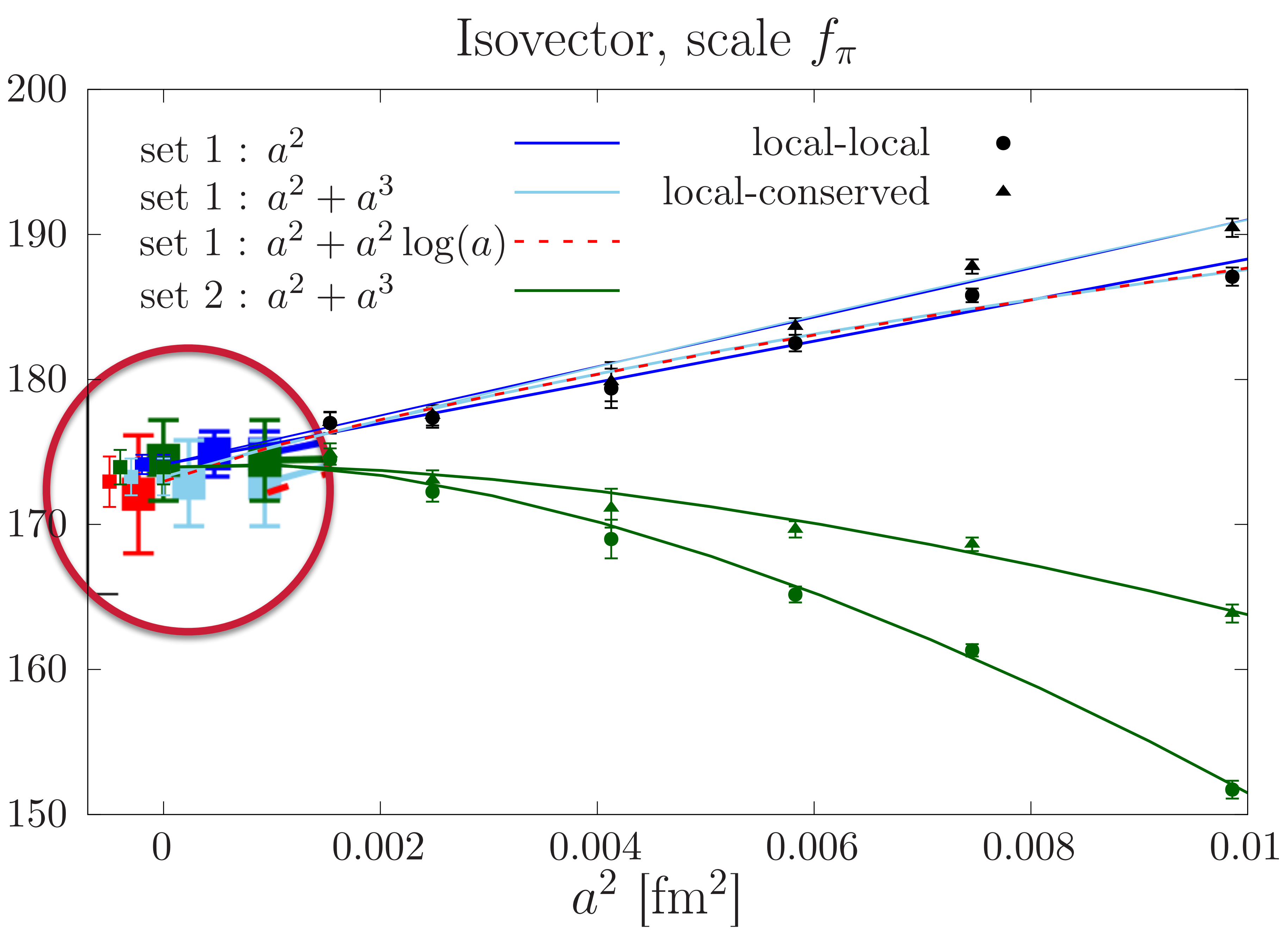
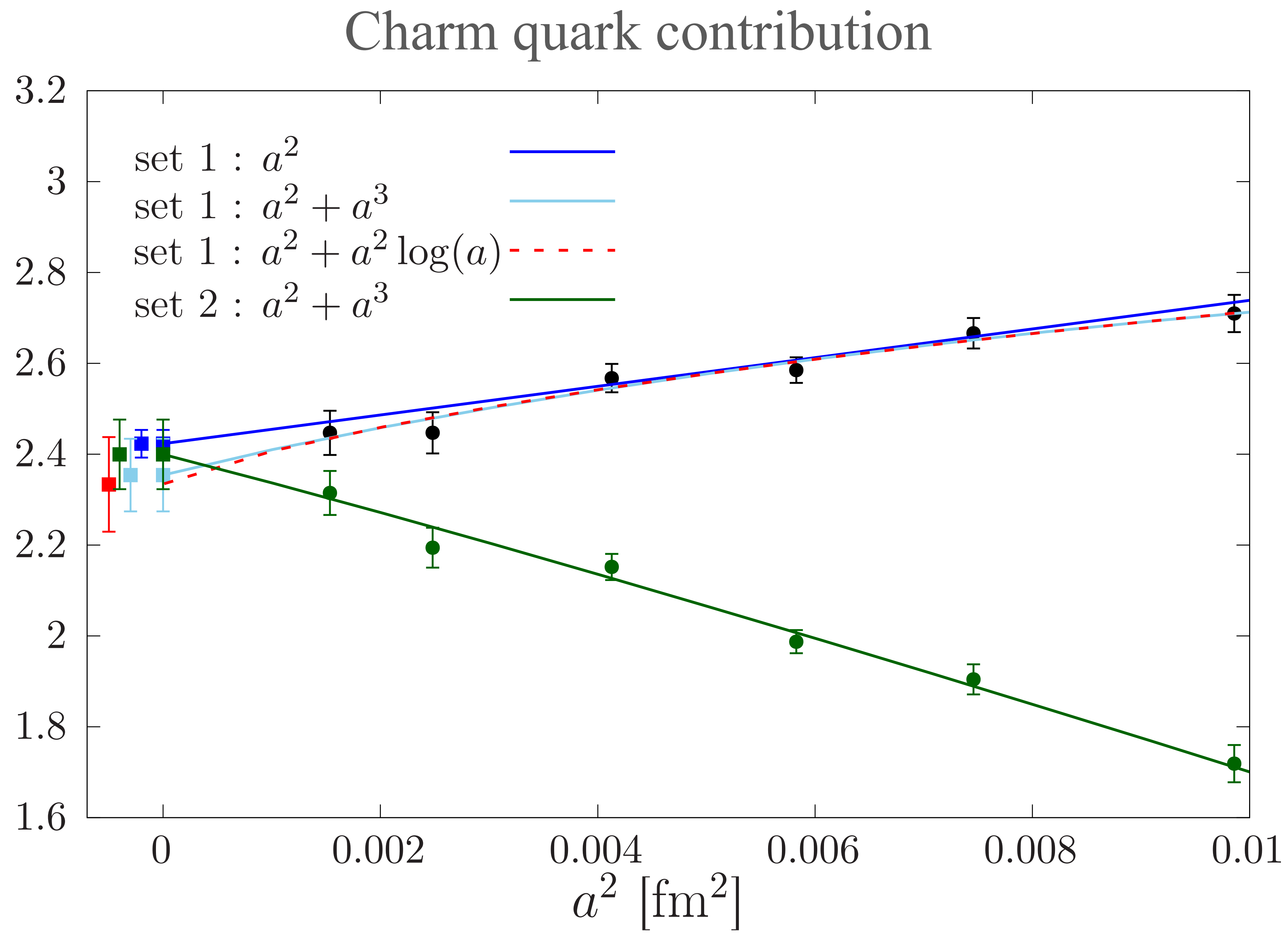
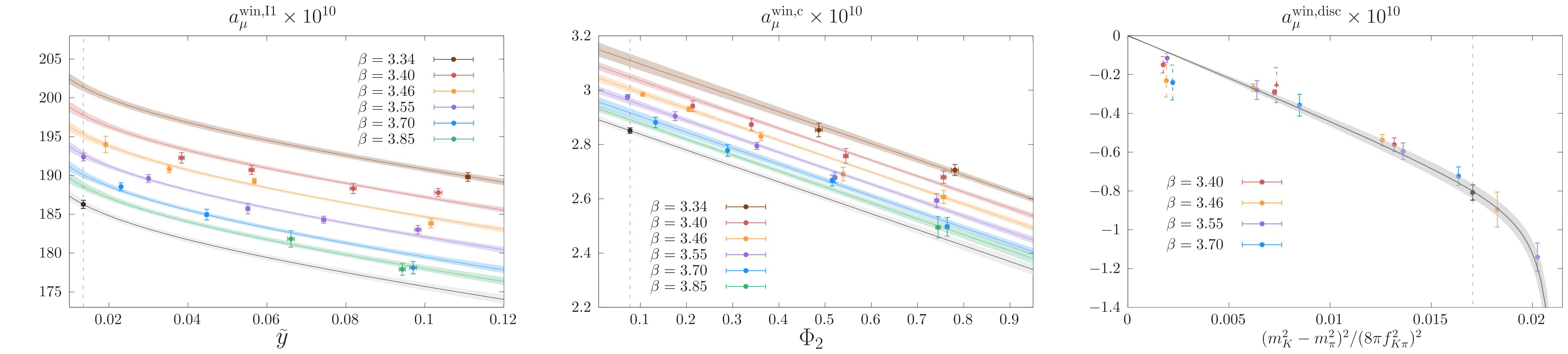
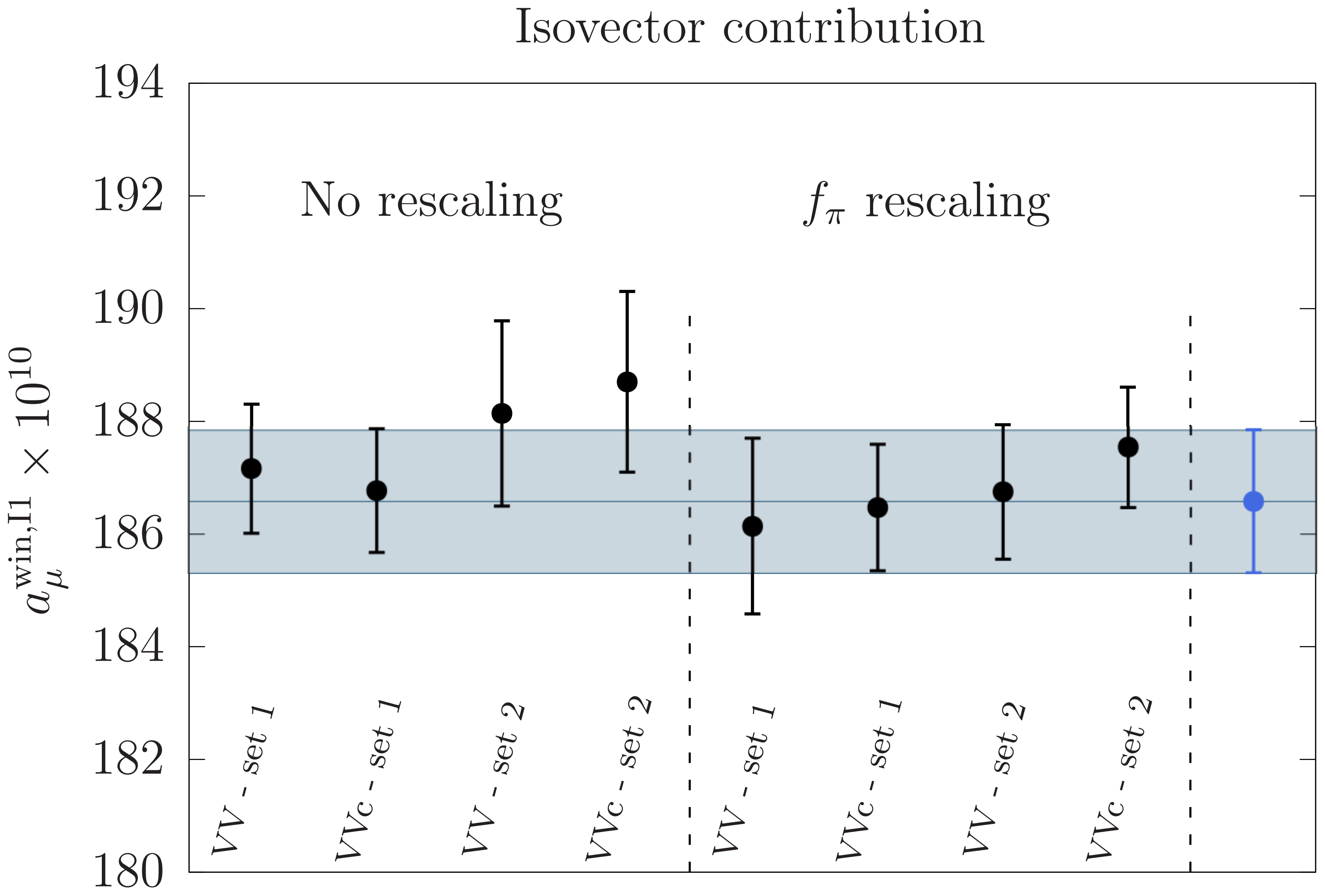




Mainz/CLS: Results at the physical point

[Cè et al., arXiv:2206.06582]





$$a_\mu^{\mathrm{win,I1}} = (186.30 \pm 0.75_{\mathrm{stat}} \pm 1.08_{\mathrm{syst}}) \times 10^{-10}\,,$$

$$a_\mu^{\mathrm{win,I0}} = a_\mu^{\mathrm{win,I0,\not{c}}} + a_\mu^{\mathrm{win,c}} = (50.30 \pm 0.23_{\mathrm{stat}} \pm 0.32_{\mathrm{syst}}) \times 10^{-10}\,,$$

$$a_\mu^{\mathrm{win,iso}} = a_\mu^{\mathrm{win,I1}} + a_\mu^{\mathrm{win,I0}} = (236.60 \pm 0.79_{\mathrm{stat}} \pm 1.13_{\mathrm{syst}} \pm 0.05_{\mathrm{Q}}) \times 10^{-10}$$

Include shift of $+(0.70 \pm 0.47) \cdot 10^{-10}$ due to isospin-breaking:

$$a_\mu^{\mathrm{win}} = (237.30 \pm 0.79_{\mathrm{stat}} \pm 1.13_{\mathrm{syst}} \pm 0.05_{\mathrm{Q}} \pm 0.47_{\mathrm{IB}}) \times 10^{-10}$$

Wilson / twisted mass fermions

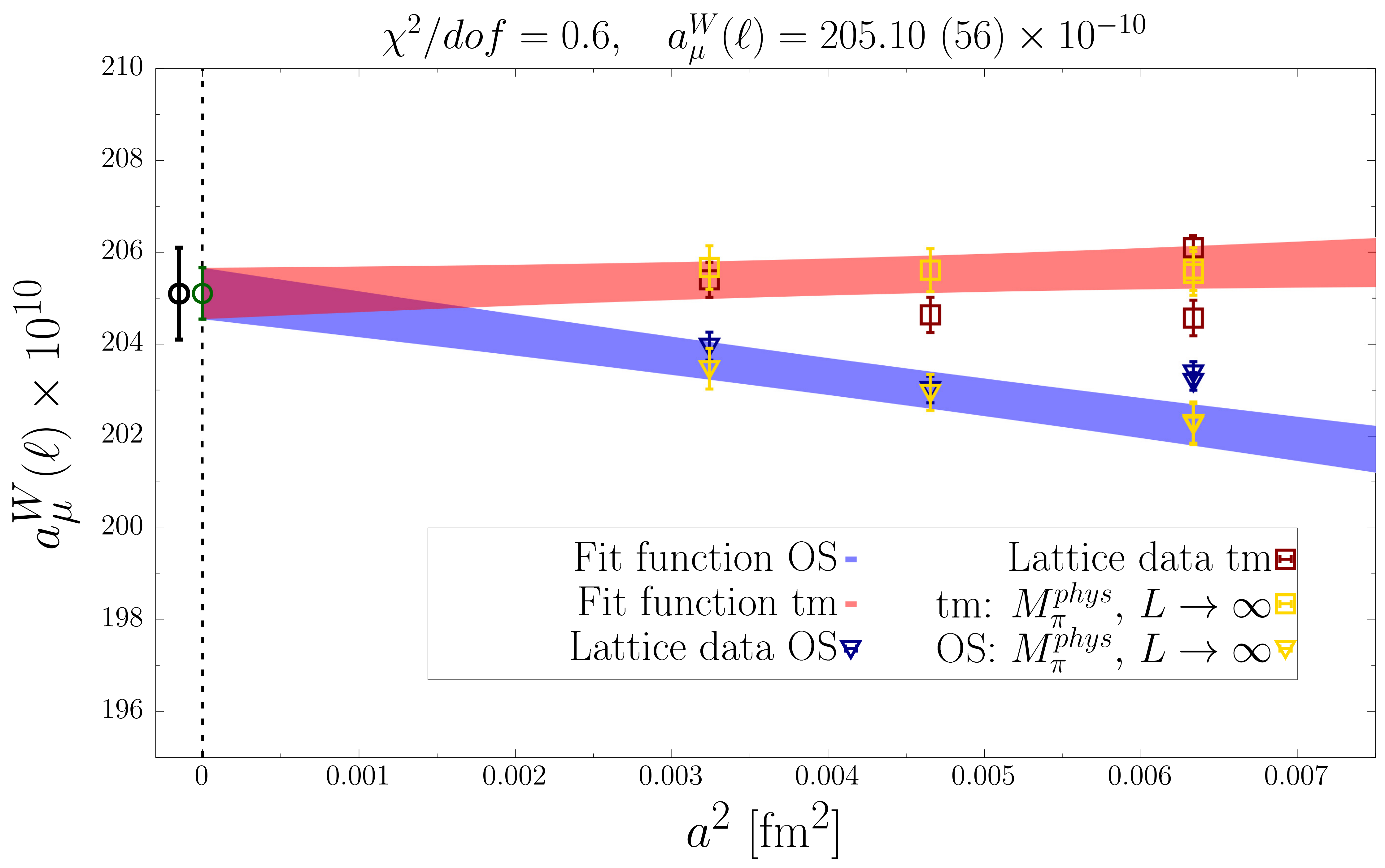
ETM Collaboration

[Alexandrou et al., arXiv:2206.15084]

- $N_f = 2+1+1$ flavour ensembles with twisted mass term: automatic $\mathrm{O}(a)$ improvement
- Three lattice spacings: $a = 0.080, 0.068, 0.057\,\mathrm{fm}$; physical pion mass
- Two discretisations of the vector current: "Osterwalder-Seiler" vs. "twisted mass"
- Results for short, intermediate and long-distance window observables in isosymmetric QCD
- Isospin-breaking corrections taken from BMW; add s, c, disconnected contributions

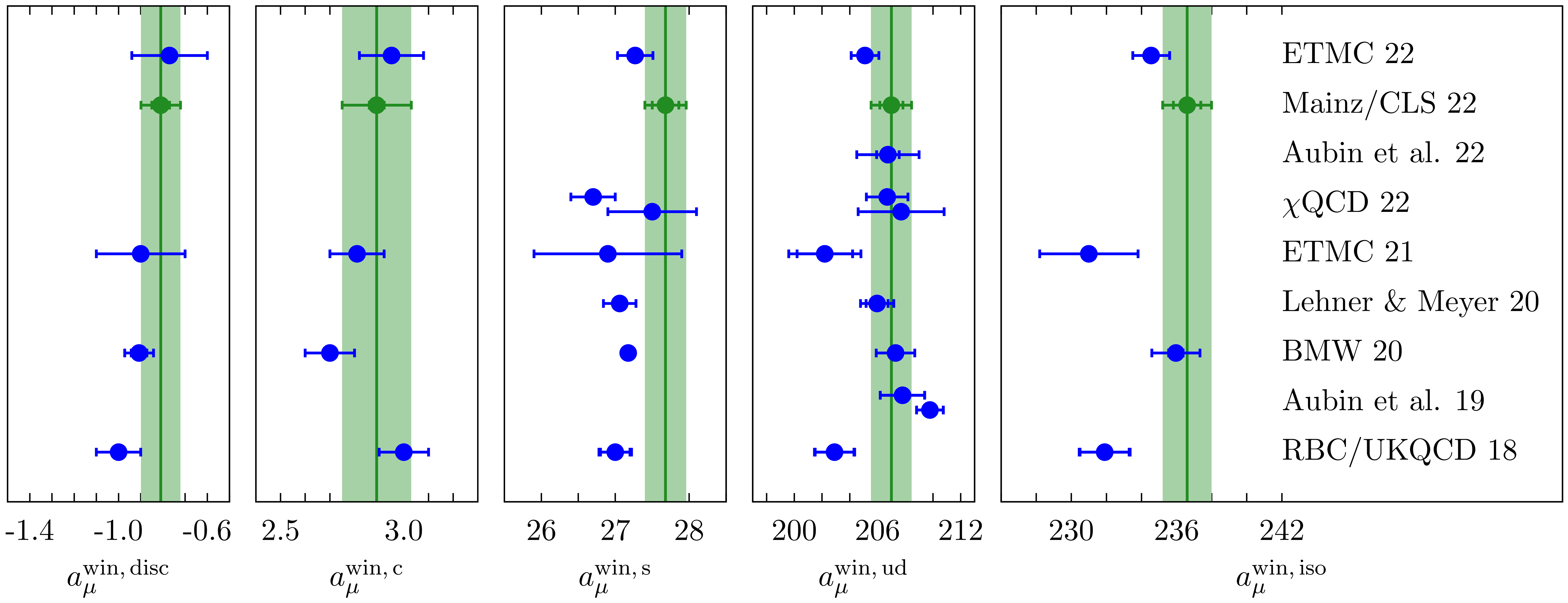
$$a_\mu^{\mathrm{win,\,ud}} = (205.1 \pm 1.0)\cdot 10^{-10}$$

$$a_\mu^{\mathrm{win}} = (235.0 \pm 1.1)\cdot 10^{-10}$$



Intermediate window observable in lattice QCD

Isosymmetric QCD:



- Broad agreement among most lattice calculations; exceptions: RBC/UKQCD 18 and ETMC 21
- ETMC 22, Mainz/CLS 21 and BMW 20 use different discretisations

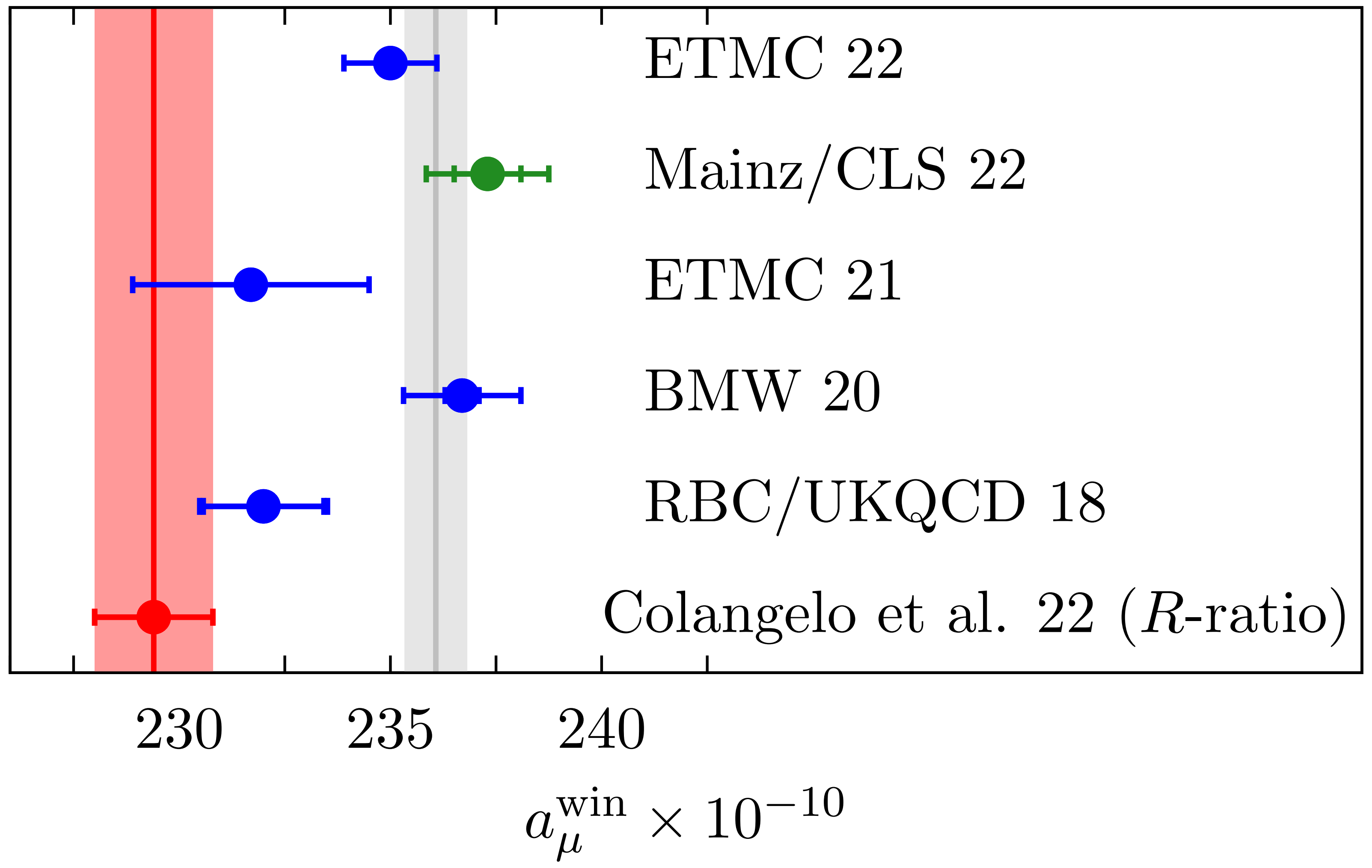
Intermediate window observable: Comparison with R-ratio

R-ratio estimate: $a_\mu^{\rm win} = (229.4 \pm 1.4) \cdot 10^{-10}$

Mainz/CLS 22: $a_\mu^{\rm win} = (237.30 \pm 1.46) \cdot 10^{-10}$

Lattice average: $a_\mu^{\rm win} = (236.08 \pm 0.74) \cdot 10^{-10}$

(ETMC 22, Mainz/CLS 22, BMW 20)



$$\Rightarrow \quad a_\mu^{\rm win}\big|_{\rm Mainz} - a_\mu^{\rm win}\big|_{R-\rm ratio} = (7.9 \pm 2.0) \cdot 10^{-10} \quad [3.9\,\sigma]$$

$$a_\mu^{\rm win}\big|_{\rm Lat-av.} - a_\mu^{\rm win}\big|_{R-\rm ratio} = (6.7 \pm 1.6) \cdot 10^{-10} \quad [4.2\,\sigma]$$

Subtract R-ratio prediction for $a_\mu^{\rm win}$ from White Paper estimate and replace by lattice average:

$$a_\mu^{\rm exp} - a_\mu^{\rm SM}\big|_{\rm Lat-av.}^{\rm win} = (18.4 \pm 5.8) \cdot 10^{-10} \quad [3.2\,\sigma]$$

Cross check: Hadronic running of α

Can the SM accommodate a higher value for a_μ without increasing the tension in the EW fit?

[Crivellin et al., 2020; Keshavarzi et al., 2020; Malaescu & Schott, 2020; Colangelo, Hoferichter, Stoffer 2020; BMW 2020/21]

Direct lattice calculation of $\Delta\alpha(-Q^2)$ on the same ensembles

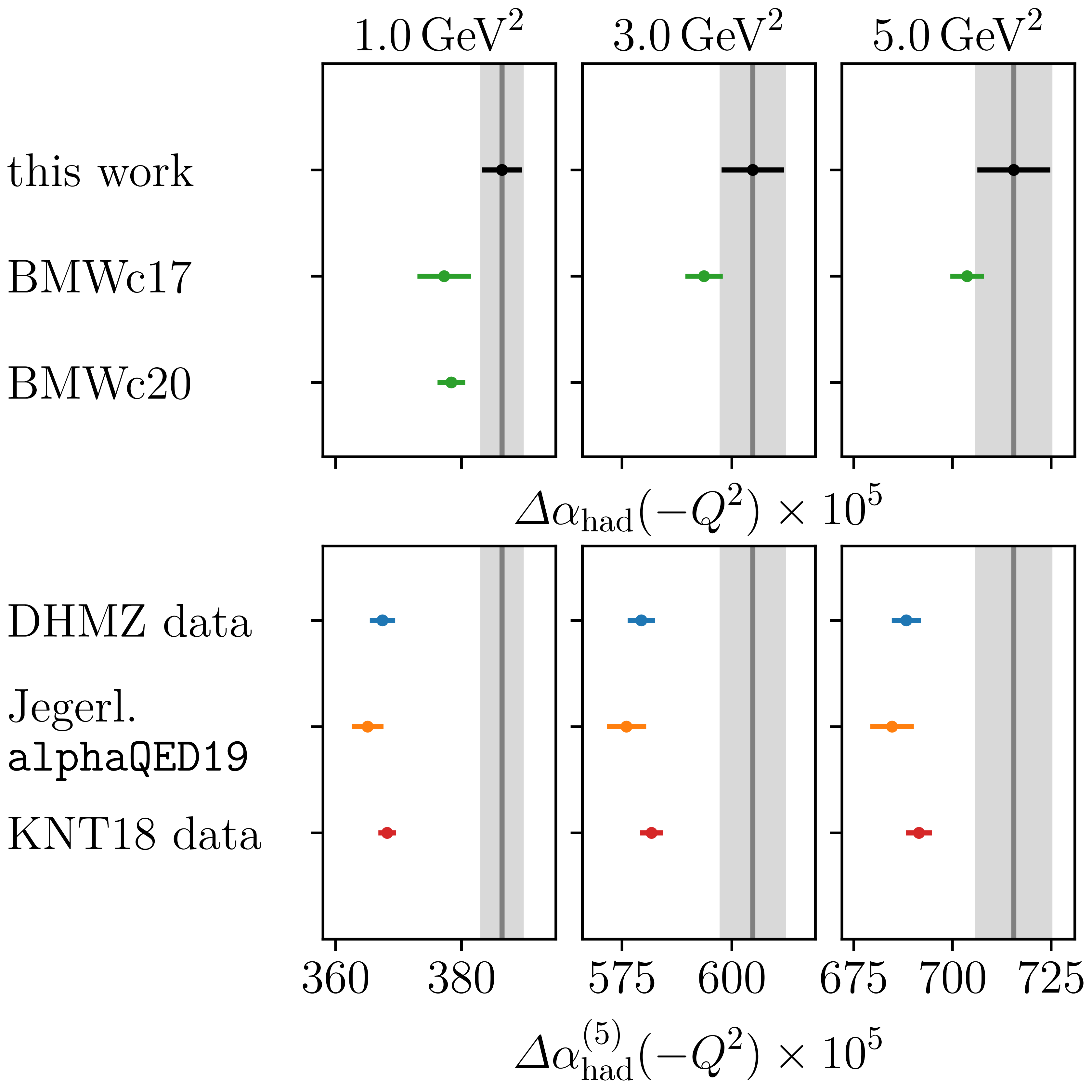
- Results broadly agree with lattice estimate by BMW
- Up to 3.5σ tension with R-ratio estimates for $Q^2 \simeq 3-7\,\mathrm{GeV}^2$

Conversion to $\Delta\alpha_{\mathrm{had}}^{(5)}(M_Z^2)$ via Euclidean split technique:

$$\Delta\alpha_{\mathrm{had}}^{(5)}(M_Z^2) = \Delta\alpha_{\mathrm{had}}^{(5)}(-Q_0^2) \quad \longleftarrow \text{Lattice QCD}$$

$$+\left[\Delta\alpha_{\mathrm{had}}^{(5)}(-M_Z^2) - \Delta\alpha_{\mathrm{had}}^{(5)}(-Q_0^2)\right]_{\mathrm{pQCD,\,disp.theory}}$$

$$+\left[\Delta\alpha_{\mathrm{had}}^{(5)}(M_Z^2) - \Delta\alpha_{\mathrm{had}}^{(5)}(-M_Z^2)\right]_{\mathrm{P.T.}}$$



[Cè et al., arXiv:2203.08676]

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Can the SM accommodate a higher value for a_μ without increasing the tension in the EW fit?

[Crivellin et al., 2020; Keshavarzi et al., 2020; Malaescu & Schott, 2020; Colangelo, Hoferichter, Stoffer 2020; BMW 2020/21]

Direct lattice calculation of $\Delta\alpha(-Q^2)$ on the same ensembles

- Results broadly agree with lattice estimate by BMW
- Up to 3.5σ tension with R-ratio estimates for $Q^2 \simeq 3-7\,\mathrm{GeV}^2$

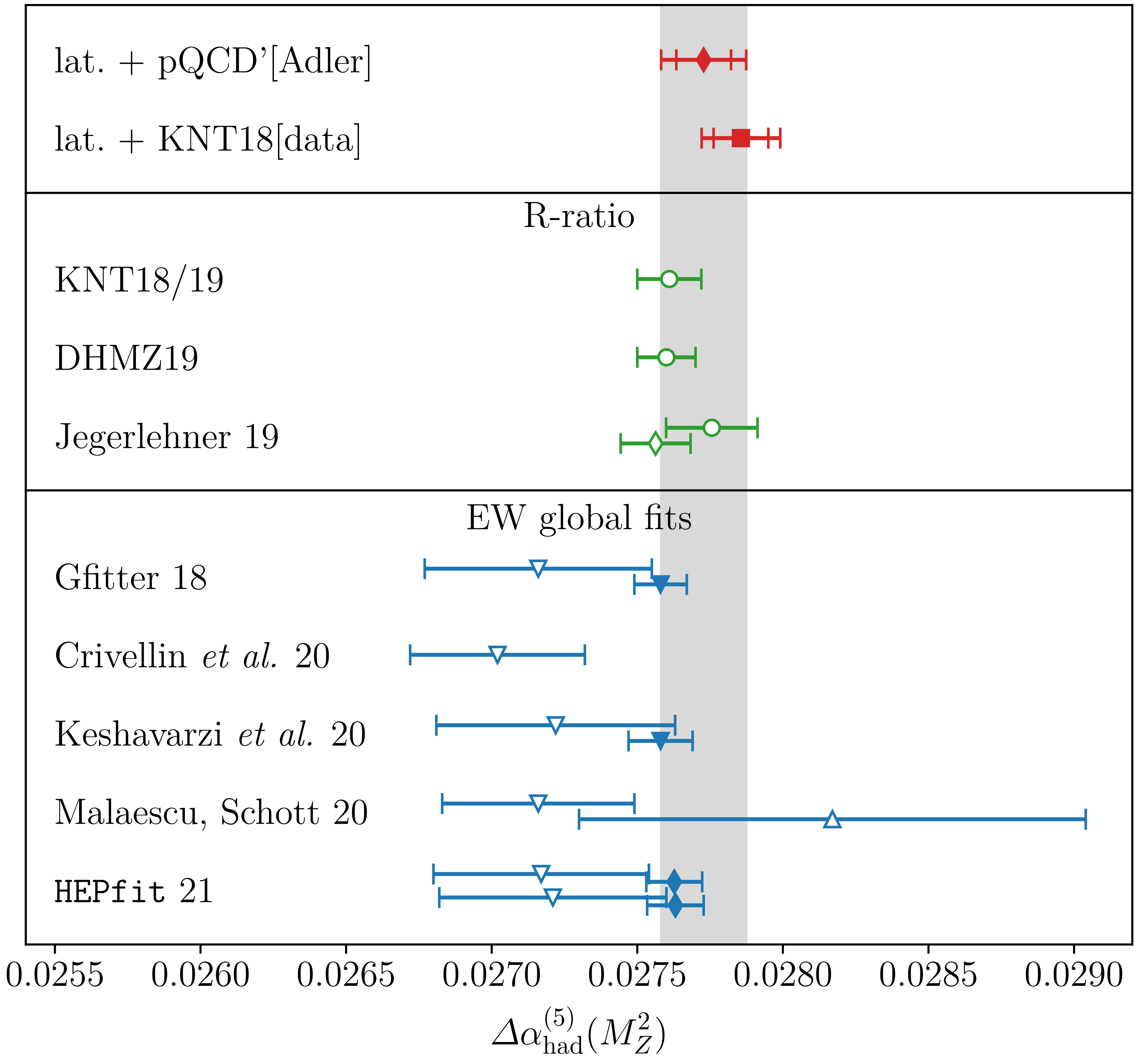
Conversion to $\Delta\alpha_{\rm had}^{(5)}(M_Z^2)$ via Euclidean split technique:

$$\Delta\alpha_{\rm had}^{(5)}(M_Z^2) = \Delta\alpha_{\rm had}^{(5)}(-Q_0^2) \quad \longleftarrow \text{Lattice QCD}$$

$$+\left[\Delta\alpha_{\rm had}^{(5)}(-M_Z^2) - \Delta\alpha_{\rm had}^{(5)}(-Q_0^2)\right]_{\rm pQCD,\,disp.theory}$$

$$+\left[\Delta\alpha_{\rm had}^{(5)}(M_Z^2) - \Delta\alpha_{\rm had}^{(5)}(-M_Z^2)\right]_{\rm P.T.}$$

$$\Rightarrow \quad \Delta\alpha_{\rm had}^{(5)}(M_Z^2) = 0.027\,73(15)$$



[Cè et al., arXiv:2203.08676]

Outlook

* Lattice calculations of the HVP contribution have reached precision level of $\lesssim 1\,\%$
* Accumulating evidence for tension between measured R-ratio and the corresponding spectral function computed on the lattice
* Different discretisations of QCD action yield results for the intermediate window that reduce the tension between SM prediction and experiment
* Window observables: benchmark quantities with reduced dependence on systematics; more finely grained intervals allow detailed cross check between lattice QCD and phenomenology
* Deviation of order $100 \cdot 10^{-11}$ between SM and experiment is a large one!

$\Rightarrow$ Precision must be increased further

Results from future runs at E989 and J-PARC E34 will be decisive!