Noncommutative (Algebraic) Geometry,

Fundamental Physics,

and Quantum Information

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Prologue : Quantum Gravity

(Geometro-)

Dynamics of Quantum Spacetime

rather than

Quantum Dynamics of (Classical)

Spacetime

Prologue : Geometrodynamics

Spacetime Geometry

Noncommutative G. $\Leftrightarrow$ Quantum Gravity

Non-Euclidean G. $\Leftrightarrow$ Classical Gravity

A hunch as a student of quantum mechanics :

Can't say the moon is there when we are not looking?!

- **no definite positions in (Newtonian) space**

  — need a better model for physical space

- **space *can only* be totality of all particle positions**

  × space → points

- **geometry with $\hat{x}$ as coordinates**

  — noncommutative geometry

Dream Comes True !

Algebraic Geometry —— Physics :-

Observable Algebra $\mathcal{A} \longleftrightarrow$ Symplectic Geometry $\mathcal{S}$

c-number picture : Commutative —— Classical

- coordinates as basic observables
- states $[\phi]$ as functionals
- Gel'fand transform : $[\phi](f) = f([\phi]) = f(x,p)|_{[\phi]}$

  $[\phi] : \mathcal{A} \longrightarrow \mathbb{R}$ — c-numbers (real numbers) value $[f]_\phi \equiv [\phi](f)$
- value of observable for a state – classical information
- evaluation map (functionals) $[\phi]$ is an algebraic homomorphism
- dynamics given by $\frac{d}{ds} f = \{f, H_s\}$, ($t$ as s)

q-number picture : Noncommutative —— Quantum +

Symplectic Geometry – (projective) Hilbert/Krein space

Observable Algebra $\mathcal{A}$ – operators β

- operator coordinates as basic observables

- states $[\phi]$ as evaluation map : algebraic homomorphism
- $[\beta]_\phi \equiv [\phi](\beta) = [f_\beta(\phi)] = \beta(\hat{x}, \hat{p})|_\phi$

  $[\phi] : \mathcal{A} \longrightarrow \mathbb{Q}$ — q-numbers noncommutative value
- value of observable for a state – quantum information

- dynamics given by $\frac{d}{ds}\beta = \{\beta, \hat{H}_s\}_q \equiv \frac{1}{i\hbar}[\beta, \hat{H}_s]$, ($t$ as s)

Evaluation as an Algebra Homomorphism :-

— real number is *only* an algebraic system

- classical $\quad [\phi] : f(x_i, p_i) \to I\!\!R \quad$ (observables have real values)

e.g. $\quad E = p^2 + x^2 = pp + xx \quad$ (1-D SHO $m = \frac{1}{2}$, $k = 2$)

$[\phi](x) = 2,\ [\phi](p) = 3 \quad \Longrightarrow$

$$[\phi](E) = [\phi](p^2) + [\phi](x^2) = [\phi](p)[\phi](p) + [\phi](x)[\phi](x) = 13$$

$$[\phi](x_i p_i) = [\phi](x_i)[\phi](p_i) = [\phi](p_i)[\phi](x_i) = [\phi](p_i x_i)$$

- **quantum** $\quad [\phi] : \beta(\hat{x}_i, \hat{p}_i) \to ?$

$$[\phi](\hat{x}_i)[\phi](\hat{p}_i) = [\phi](\hat{p}_i)[\phi](\hat{x}_i) + [\phi](i\hbar\hat{I})$$

$\Longrightarrow [\phi](\beta(\hat{x}_i, \hat{p}_i))$ **has to be a noncommutative algebra**

Intuitive Concepts *are not* Classical !

– Quantum Concepts *no less Intuitive*

– the Classical ones *only more familiar*

the main culprit : 'Quine's *convenient fiction*' – real numbers

The New Convenient Fiction —— q-numbers :-

- $[\beta]_\phi = \{ f_\beta|_\phi, V_{\beta_n}|_\phi \}$; $f_\beta(z_n, \bar{z}_n) = \frac{\langle\phi|\beta|\phi\rangle}{\langle\phi|\phi\rangle}$, $|\phi\rangle = \sum_n z_n |n\rangle$,

$$V_{\beta_n} = \frac{\partial f_\beta}{\partial z_n} = -f_\beta \bar{z}_n + \sum_m \bar{z}_m \langle m|\beta|n\rangle$$

Cirelli et.al 90

- Kähler product : $[\beta\gamma]_\phi = [\beta]_\phi \star_\kappa [\gamma]_\phi$

$$f_{\beta\gamma} = f_\beta f_\gamma + \sum_n V_{\beta_n} V_{\gamma_{\bar{n}}} \ , \quad V_{\beta\gamma_n} = -f_{\beta\gamma} \bar{z}_n + \sum_{m,l} \bar{z}_m \langle m|\beta|l\rangle\langle l|\gamma|n\rangle$$

Deutsch & Hayden 00

- locality of quantum information from/in Heisenberg picture

Galvão & Hardy 03

- Substituting a Qubit for an Arbitrarily Large Number of Classical Bits'

- f_β Kählerian – Hamiltonian flows as isometries

Classical Mechanics as Symplectic Geometry :-

• Eq. of motion : $\frac{d}{dt}x^i = \frac{\partial H}{\partial p^i}$, $\frac{d}{dt}p^i = -\frac{\partial H}{\partial x^i}$

• symplectic/Poisson structure :

$$\{f(x^i, p^i), H_\sigma\} = \sum_i \left(\frac{\partial f}{\partial x^i}\frac{\partial H_\sigma}{\partial p^i} - \frac{\partial f}{\partial p^i}\frac{\partial H_\sigma}{\partial x^i} \right)$$

• from geometry (coordinates indep.) $\frac{d}{d\sigma} f = \{f, H_\sigma\}$

$\longrightarrow$ f values on curves of constant H_σ for each value of σ

— H_σ **generates symmetry transformations of geometry**

• **dynamics : energy function → phase sp same at all time**

Symplectic Geometry – Quantum Dynamics :-

- (Hamilton-)Schrödinger : $\frac{d}{d\sigma} f = \{f, f_{\beta_\sigma}(z_n, \bar{z}_n)\}$

  — $\beta_t \equiv \hat{H} = \frac{\hat{p}_i \hat{p}^i}{2m} + V(\hat{x}_i)$

- $i\hbar \frac{d}{dt} |\phi\rangle = \hat{H} |\phi\rangle \qquad \Rightarrow \qquad \frac{dq_n}{dt} = \frac{\partial H}{\partial s_n} \,, \qquad \frac{ds_n}{dt} = -\frac{\partial H}{\partial q_n}$

  $$z_n = \frac{1}{\sqrt{2}}(q_n + i s_n) \,, \qquad H(z_n, \bar{z}_n) = \frac{1}{\hbar} \langle\phi| \hat{H} |\phi\rangle$$

- Heisenberg : $\frac{d}{d\sigma}\beta = \{\beta, \beta_\sigma(\hat{x}^i, \hat{p}^i)\}_q = \frac{1}{i\hbar}[\beta, \beta_\sigma]$

- $\frac{d\hat{x}_i}{dt} = \frac{\partial \hat{H}}{\partial \hat{p}_i} \,, \quad \frac{d\hat{p}_i}{dt} = -\frac{\partial \hat{H}}{\partial \hat{x}_i} \qquad \Longleftarrow \frac{\partial}{\partial \hat{x}_i} = -\frac{1}{i\hbar}[\hat{p}_i, \cdot] \,, \quad \frac{\partial}{\partial \hat{p}_i} = \frac{1}{i\hbar}[\hat{x}_i, \cdot]$

Schrödinger Vs Heisenberg – **coordinate transformation** :-

- coordinate map: $\hat{f} : (q_n, s_n) \longrightarrow (f_{\hat{x}_i}, f_{\hat{p}_i}) \rightarrow (\hat{x}_i, \hat{p}_i)$

- intuitively : $\hat{x}_i$, $\hat{p}_i$ as coordinates of quantum phase space

- **noncommutative differential symplectic geometry** :

$$f_\beta \ , \ df_\beta = f_{d\beta} \ , \ d\beta \ , \ \mathcal{X}_\beta = \frac{1}{i\hbar}[\cdot, \beta] \ , \ \ldots \quad \textbf{as pull-backs}$$

$$\frac{d\cdot}{dt} \longrightarrow d\cdot \qquad (\langle\delta\phi|\beta|\phi\rangle + \langle\phi|\beta|\delta\phi\rangle) = \langle\phi|d\beta|\phi\rangle$$

$$|\delta\phi\rangle = dz_n \, |n\rangle \text{ with constant } \langle\phi|\phi\rangle$$

$$d\beta = (d\hat{x}^i \frac{\partial}{\partial\hat{x}^i} + d\hat{p}^i \frac{\partial}{\partial\hat{p}^i})\beta = [D, \beta] \ , \qquad d\eta = D\eta - (-1)^k \eta D \ ,$$

$$d^2 = 0 \ , \qquad \beta^\dagger = \bar{\beta} \ , \qquad (d\eta)^\dagger = (-1)^k d(\eta^\dagger) \ , \quad (\eta\eta')^\dagger = \bar{\eta}'\bar{\eta}$$

Phase Space as Physical Space(time) :-

- **no noncommutativity with $\hat{x}^i$ only**
- complex phase rotation on $\boldsymbol{z^n \longrightarrow e^{i\theta}(q^n + is^n)}$

— **mix configuration & momentum variables**

- **quantum relativity symmetry** — $e^{i\theta}$ generated by $[\hat{x}^i, \hat{p}^i]$

— **no other irreducible representation**

- **phase space metric important**

$\longrightarrow \boldsymbol{dx^2 + dp^2}$ — proper **units** $\boldsymbol{x \gg 1}$ and $\boldsymbol{p \ll 1}$

'spacetime' & energy-momentum $\longrightarrow$ SPACETIME

(Einstein : space & time → 'spacetime')

Quantum Mechanics

can and should be seen as

Particle Dynamics on the Quantum Space

rather than

Quantized Dynamics on the Newtonian space

Answering *Einstein*-Bohr :-

- **Noncommutative Reality** – **definite q-number values**

— ***no probability***, Heisenberg uncertainty – $\sum_n |V_{\beta_n}|^2$

— ***experimentally accessible (?)***

- **no hidden variables**, *but* **'hidden values'** of variables

- **fully intuitive QM**, local quantum information

- ***dynamics as geometry***, from relativity symmetry

— **noncommutativity as curvature**

$H_R(3)$ / $H_R(1,3)$ Symmetry Theoretical Formulation :-

$$[K_i, P_j] = i\hbar\delta_{ij} M \ ,$$

$$[J_{ij}, J_{hk}] = i\hbar(\delta_{jk} J_{ih} + \delta_{ih} J_{jk} - \delta_{ik} J_{jh} - \delta_{jh} J_{ik})$$

$$[J_{ij}, K_k] = i\hbar(\delta_{ik} K_j - \delta_{jk} K_i) \ , \quad [J_{ij}, P_k] = i\hbar(\delta_{ik} P_j - \delta_{jk} P_i) \ .$$

- $\hat{x}_i = \frac{1}{m}\hat{K}_i \leftarrow \hat{M}^{-1}\hat{K}_i$, **mass as Casimir invariant**

— $\hat{x}_i \equiv \frac{1}{m}\hat{K}_i = \frac{1}{m}(\hat{K}_{ia} \otimes \hat{I} + \hat{I} \otimes \hat{K}_{ib}) = \frac{1}{m}(m_a \hat{X}_{ia} \otimes \hat{I} + \hat{I} \otimes m_b \hat{X}_{ib})$

- **no time (translation) from relativity symmetry**

- irr. rep. as components of regular rep. (only phase spaces)
  — coherent state ↔ Moyal star-product

- as module of Weyl algebra – **Gel'fand-Kirillov dim. 3**

Summary of Symmetry Theoretical Formulation :-

- irr. rep. $\longrightarrow$ **Hilbert / Krein space of states**
- universal enveloping algebra / group C^*-algebra
  $\longrightarrow$ **observable algebra (functions of basic observables)**

- (commutator) $\longrightarrow$ Poisson bracket $\longrightarrow$ **Hamiltonian dynamics**
  $\updownarrow$
- **group invariant inner product** $\longrightarrow$ (Kähler) structure
- **Euclidean / Minkowski metric operator**
- phase space as **space(time)**

★ **contraction gives app. theory**

Fundamental (Special) Quantum Relativity:

$SO(2,4) \sim SU(2,2)$ (*cf.* deformed S.R.)

- contains noncommuting X_μ and P_μ , I
- contains Lorentz symmetry $J_{\mu\nu}$
- stable symmetry, no deformation
- G, $\hbar$, c in structural constants

$$\begin{array}{ccccc} SO(2,4) \longrightarrow H_R(1,3) & \longrightarrow & H_{GH}(3) \supset \tilde{G}(3) \supset H_R(3) & \\ \downarrow & (\frac{1}{c^2} \to 0) & \downarrow & (\hbar \downarrow 0) \\ ISO(1,3) \subset S(1,3) & \longrightarrow & S_G(3) \supset G(3) \supset S(3) & \end{array}$$

- (Lie algebra) contractions as approximations
- deformation is 'inverse' of contraction

Lorentz Covariant Quantum Mechanics/ Spacetime

- quantum relativity $H_R(1,3)$

  — Y_μ , P_μ , M , $J_{\mu\nu}$ $(\hat{Y}_\mu = m\hat{x}_\mu\ \hat{J}_{\mu\nu} = \hat{x}_\mu\hat{p}_\nu - \hat{x}_\nu\hat{p}_\mu + \hat{S}_{\mu\nu})$

- pseudo-unitary rep. as in Minkowski spacetime

  — $\hat{x}_\mu = x_\mu \star = x_\mu + i\partial_{p^\mu}, \quad \hat{p}_\mu = p_\mu \star = p_\mu - i\partial_{x^\mu}$

- Krein space ${}_\eta\langle\cdot|\cdot\rangle = \langle\cdot|\hat{\eta}|\cdot\rangle \quad \leftrightarrow \quad \hat{\eta}\,\hat{x}_\mu\hat{\eta}^{-1} = \hat{x}^\mu$

  — well-defined integrals for wavefunctions

- noncommutative Minkowski spacetime

- $c \to \infty$ limit gives $H_R(3)$ QM at each time value

What Next ? :-

- practical ? **quantum information as q-numbers**

- Math : **q-numbers as scalars ?** – algebra; tangent space

  — **Nullstellensatz** (complete factorization)

- **theory with noncommutative X-X, P-P ?**

  — **quantum phase space as minimal NCG**

- **quantum field theory ?** — $m = 0$

- **gravity?** – geometrodynamics of noncommutative spacetime

THANK YOU !