langevin simulation of dark matter kinetic equilibration[1,2]

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[2] based on Seyong Kim & ML, 2302.05129

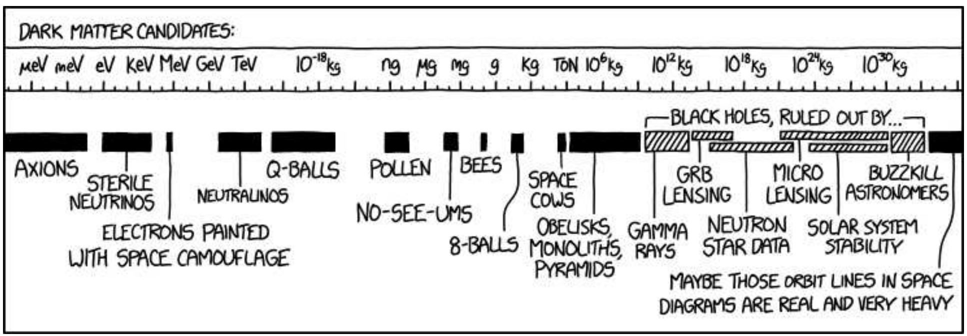
what is dark matter?

yet to be discovered particles? basic requirements:

- not visible $\Rightarrow$ electrically neutral
- around long ago & still today $\Rightarrow$ stable or very long-lived
- correct structure formation long ago $\Rightarrow$ rather heavy

known particles fail to satisfy these requirements

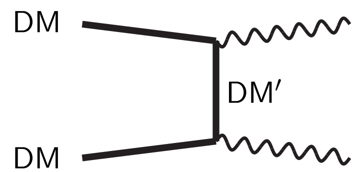
cartoon candidates from `https://xkcd.com/2035/`:



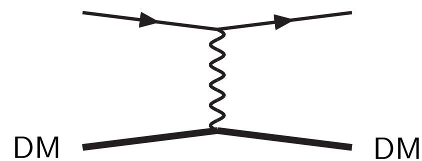
very influential: “wimp paradigm”

postulate the existence of weakly interacting massive particles (“heavy neutrinos”) which cannot decay and are thus stable

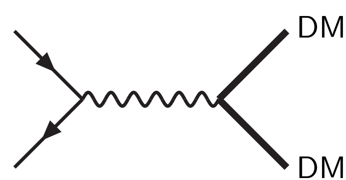
“indirect detection” from galactic center:



“direct detection” by nuclear recoil:



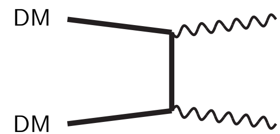
“collider search” through missing energy:



text-book wimp is in trouble

lee-weinberg equation[3] ($n =$ number density, $H =$ Hubble rate)

$$(\partial_t + 3H)n = -\langle \sigma v_{\text{rel}} \rangle \, (n^2 - n_{\text{eq}}^2)$$



linearize around equilibrium:

$$n = n_{\text{eq}} + \delta n \,, \quad n^2 - n_{\text{eq}}^2 \approx 2 n_{\text{eq}} \delta n$$

parametrize cross section:

$$\langle \sigma v_{\text{rel}} \rangle \equiv \frac{\alpha^2}{M^2} \,, \quad M \equiv M_{\text{DM}}$$

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[3] B.W. Lee and S. Weinberg, *Cosmological Lower Bound...*, Phys. Rev. Lett. 39 (77) 165

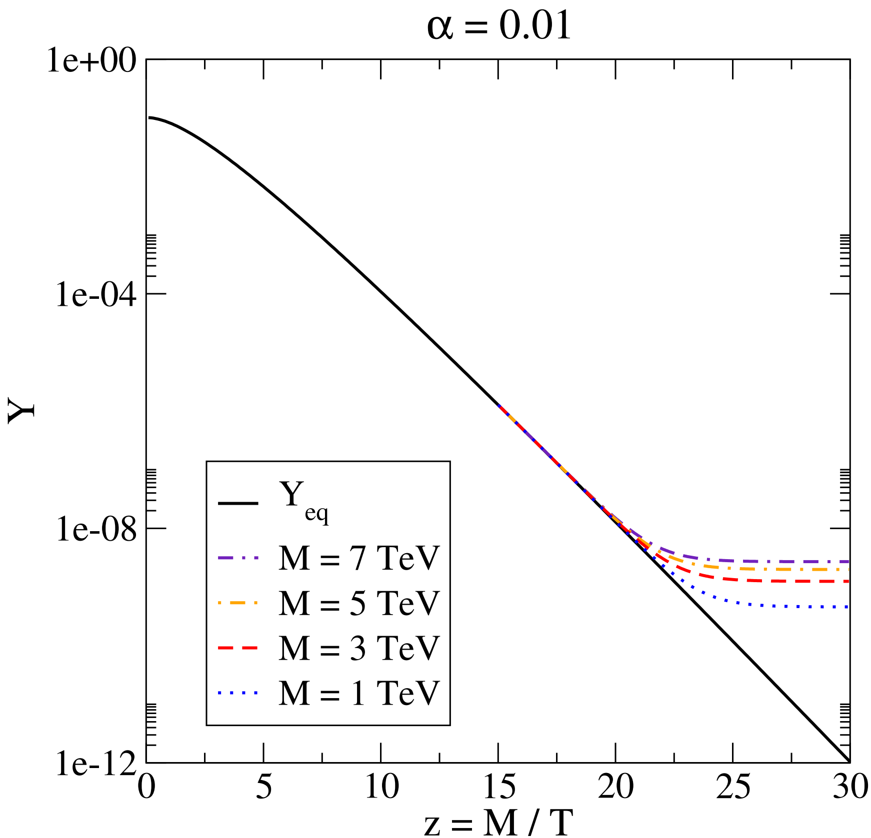
$$\Rightarrow \quad \boxed{(\partial_t + 3H)n \approx -\frac{2\alpha^2 n_{\text{eq}}}{M^2}\,\delta n}$$

equilibrium number density is a known function of T, M:

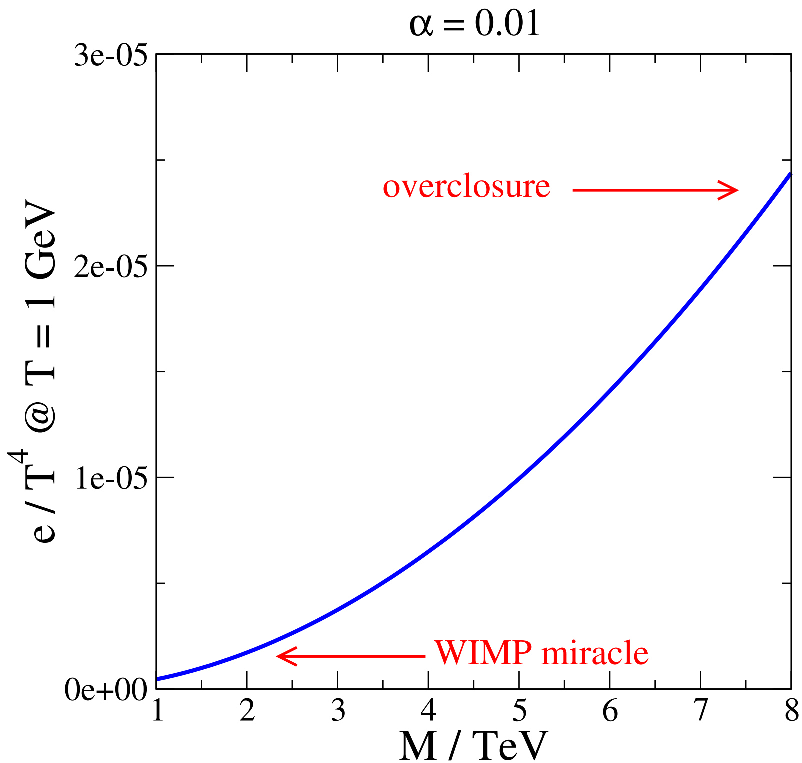
$$n_{\text{eq}} \propto \int \frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^3} \frac{1}{e^{\sqrt{p^2+M^2}/T} \pm 1} \approx \left(\frac{MT}{2\pi}\right)^{3/2} e^{-M/T}$$

the right-hand side becomes very small if $\alpha^2 n_{\text{eq}}/M^2 \ll H$

indeed a numerical solution shows a “freeze-out” ($Y \equiv n/s$):



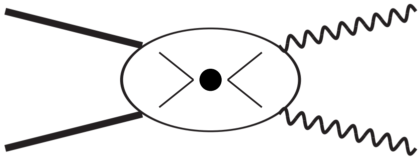
$\Rightarrow$ final energy density ($e \equiv Mn$) grows faster than $\sim M$:



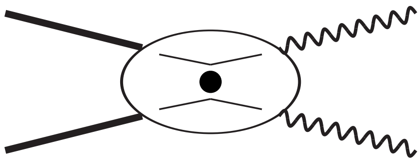
null searches at LHC push up M, so danger of "overclosure"

could increased $\langle \sigma v_{\text{rel}} \rangle$ help?

large cross section could originate via “resonant” effects

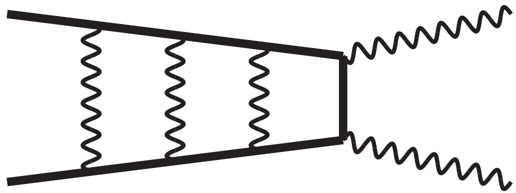


s-channel



t-channel

simplest t-channel enhancement:[4] "sommerfeld effect"[5]



$$\langle \sigma v_{\text{rel}} \rangle \longrightarrow \langle \sigma_{\text{tree}}\, v_{\text{rel}}\, S(v_{\text{rel}}) \rangle$$

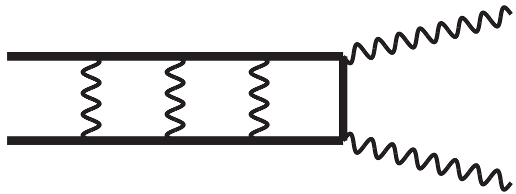
for attractive coulomb-like interaction:

$$S(v_{\text{rel}}) \sim \frac{\alpha}{v_{\text{rel}}} \quad \text{for} \quad v_{\text{rel}} \lesssim \alpha$$

[4] e.g. J. Hisano *et al*, *Non-perturbative effect on ... dark matter*, hep-ph/0610249

[5] e.g. L.D. Landau and E.M. Lifshitz, *Quantum Mechanics*, Third Edition, §136

even more efficient:[6] bound states



$$M_{\text{bound}} = 2M - \Delta E \;\Rightarrow\; e^{-M_{\text{bound}}/T} > e^{-2M/T}$$

$\Rightarrow$ exponential enhancement $e^{\Delta E/T}$ over the tree-level estimate

(typically the dark sector contains several species, DM and DM', and perhaps only one of them forms bound states)

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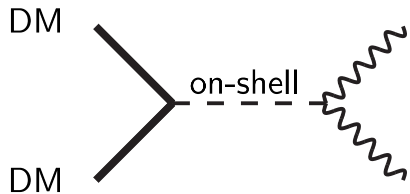
[6] e.g. B. von Harling and K. Petraki, *Bound-state formation for ...*, 1407.7874

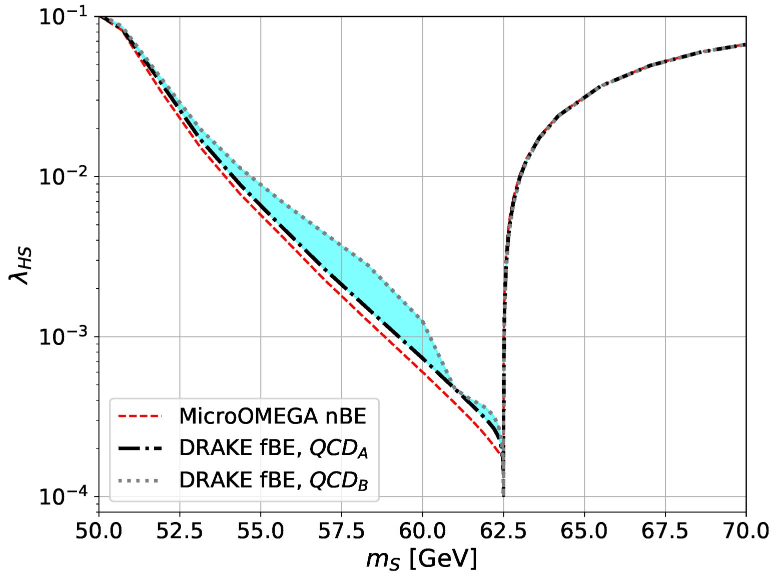
the boost can also be after annihilation: s-channel



$$M_{\text{resonance}} = 2M + \Delta M \;\Rightarrow\; \langle \sigma v_{\text{rel}} \rangle$$ “diverges” on-shell

$\Rightarrow$ a very large cross section?

example of phenomenology from a higgs resonance[7]



⇒ thanks to a large cross section, very small couplings allowed

[7] M. Di Mauro, C. Arina, N. Fornengo, J. Heisig and D. Massaro, *Dark matter at the Higgs resonance*, 2305.11937

why the large uncertainties?

given the peculiar dynamics, the usual assumption of kinetic equilibrium has been questioned for s-channel resonances[8]

note that kinetic equilibrium is certainly **not** there for “freeze-in” dark matter — here we focus on non-relativistic “freeze-out” case

in addition there appear to be large QCD uncertainties

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[8] T. Binder, T. Bringmann, M. Gustafsson and A. Hryczuk, *Early kinetic decoupling of dark matter: when the standard way of calculating the thermal relic density fails*, 1706.07433; K. Ala-Mattinen and K. Kainulainen, *Precision calculations of dark matter relic abundance*, 1912.02870; T. Binder, T. Bringmann, M. Gustafsson and A. Hryczuk, *Dark matter relic abundance beyond kinetic equilibrium*, 2103.01944; T. Abe, *Early kinetic decoupling and a pseudo-Nambu-Goldstone dark matter model*, 2106.01956; K. Ala-Mattinen, M. Heikinheimo, K. Kainulainen and K. Tuominen, *Momentum distributions of cosmic relics: Improved analysis*, 2201.06456

how to study kinetic non-equilibrium?[9]

9 literature is based on boltzmann equations, but then it is not clear how to address the other uncertainty, from NLO or non-perturbative QCD effects

rare interactions & non-relativistic limit: langevin equation

$$\dot{p}^i = -(\eta + H)p^i + f^i \,, \quad \langle\, f^i(t_1)\, f^j(t_2)\,\rangle = \zeta\, \delta^{ij}\, \delta(t_1 - t_2)$$

fluctuation-dissipation relation $\Rightarrow$ there is only one free coupling

$$\underbrace{\eta = \frac{\zeta \langle \mathbf{v}^2 \rangle}{6T^2}}_{\text{defines temperature}} \quad , \qquad \underbrace{\langle \mathbf{v}^2 \rangle \approx \frac{3T}{M}}_{\text{defines kinetic mass}}$$

hubble expansion can be hidden with co-moving variables

$$x \equiv \ln\left(\frac{T_{\mathsf{max}}}{T}\right) , \quad (\ldots)' \equiv \frac{\mathrm{d}(\ldots)}{\mathrm{d}x}$$

with entropy density, speed of sound, and hubble rate:

$$\hat{p}^{i} \equiv \frac{p^{i}}{s^{1/3}} , \quad \hat{\eta} \equiv \frac{\eta}{3c_s^2 H} , \quad \hat{\zeta} \equiv \frac{\zeta}{3c_s^2 H\, s^{2/3}}$$

this yields the dimensionless evolution equations

$$(\hat{p}^{i})' = -\hat{\eta}\,\hat{p}^{i} + \hat{f}^{i} , \quad \langle\, \hat{f}^{i}(x_1)\, \hat{f}^{j}(x_2)\, \rangle = \hat{\zeta}\,\delta^{ij}\,\delta(x_1 - x_2)$$

where $\hat{\eta}$ and $\hat{\zeta}$ are not constant but evolve rapidly with x

computation of ζ in quantum field theory

consider the real-time 2-point correlator of the force

force = time derivative of the spatial components of a current

afterwards, model-dependent but weakly coupled fields (dark matter, mediator) can be "integrated out" perturbatively

left over is a correlation function of strongly coupled objects (QCD currents composed of quarks and gluons)

parametrization: $\zeta \;\equiv\; \dfrac{\xi\, T^7}{(100\ \text{GeV})^4}$

example: scalar singlet model[10]

$$\mathcal{L} \equiv \mathcal{L}_{SM} + \left\{ \frac{1}{2}\partial^\mu \varphi \, \partial_\mu \varphi - \left[\frac{1}{2} \left(m_{\varphi 0}^2 + \kappa \, \phi^\dagger \phi \right) \varphi^2 + \frac{1}{4} \lambda_\varphi \, \varphi^4 \right] \right\}$$

introduce an effective non-relativistic field ψ as

$$\varphi \simeq \frac{1}{\sqrt{2 m_\varphi}} \left(\psi \, e^{-i m_\varphi t} + \psi^* \, e^{i m_\varphi t} \right)$$

ψ has a conserved particle number current, broken by interactions

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[10] V. Silveira and A. Zee, *Scalar Phantoms*, PLB 161 (1985) 136; J. McDonald, *Gauge singlet scalars as cold dark matter*, hep-ph/0702143; C.P. Burgess, M. Pospelov and T. ter Veldhuis, *The Minimal Model of nonbaryonic dark matter: a singlet scalar*, hep-ph/0011335; J.M. Cline, K. Kainulainen, P. Scott and C. Weniger, *Update on scalar singlet dark matter*, 1306.4710

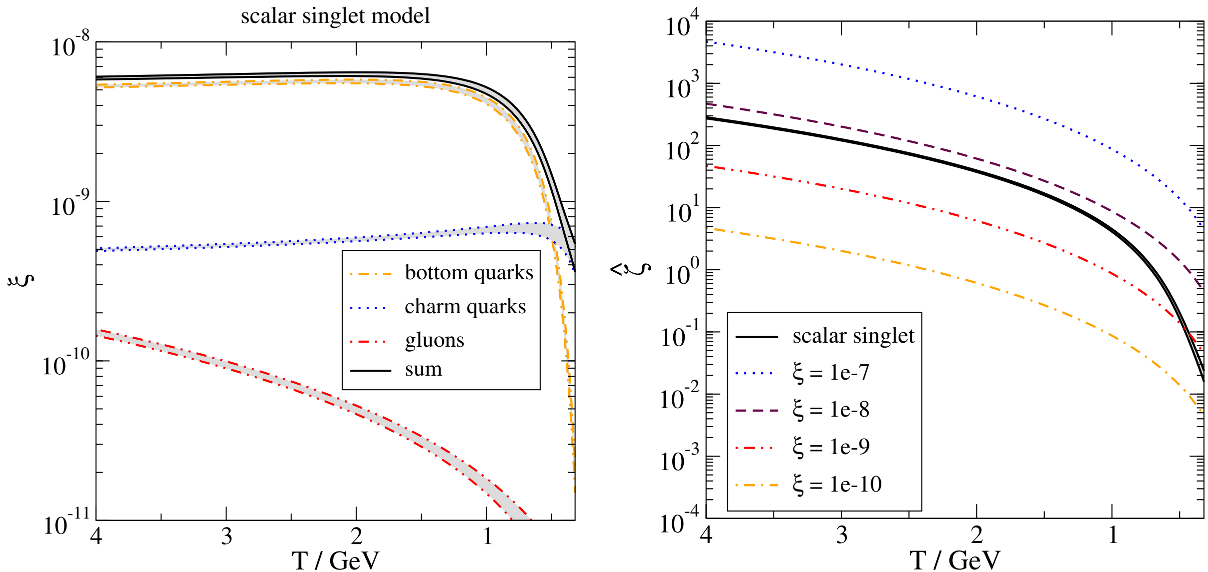
after the dust settles, quark contribution in higgs phase

$$\mathcal{L}_{SM} \supset -(\text{higgs}) \frac{h_b \bar{b} b}{\sqrt{2}}$$

$$\zeta \supset \frac{4\kappa^2 m_b^2 N_c T}{3\pi^3 m_\varphi^2 m_h^4} \int_{m_b}^{\infty} \mathrm{d}\epsilon \, \epsilon^3 (\epsilon^2 - m_b^2) \, n_{\mathrm{F}}(\epsilon)$$

$$\leq \frac{31\pi^3 \kappa^2 m_b^2 N_c T^7}{189 m_\varphi^2 m_h^4} \approx \frac{\xi \, T^7}{(100 \text{ GeV})^4}$$

numerical examples of ξ and $\hat{\zeta}$



in the following consider the four different ξ curves

simulations and results

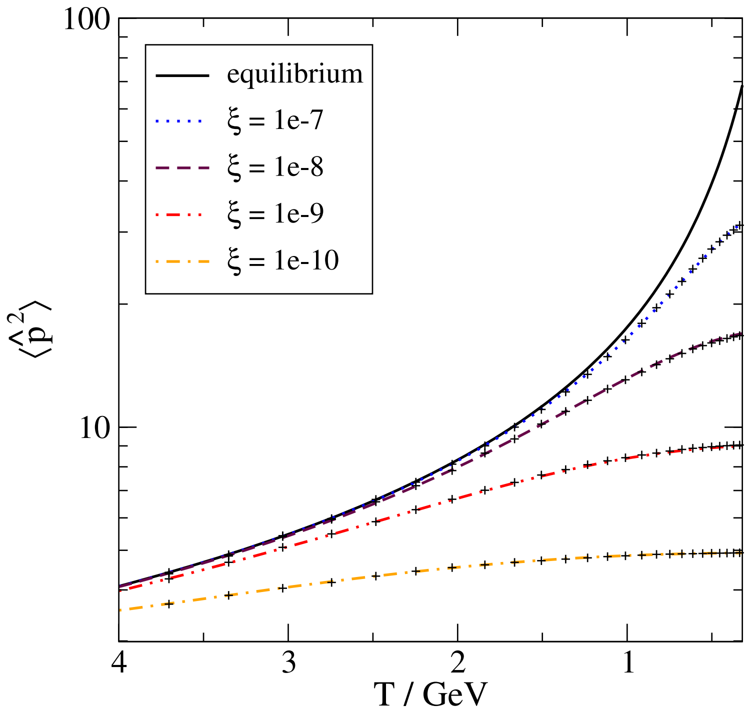
analytic solution for the second moment

$$(\hat{p}^{i})' \;=\; -\hat{\eta}\,\hat{p}^{i} + \hat{f}^{i}\;, \quad \langle\, \hat{f}^{i}(x_1)\, \hat{f}^{j}(x_2)\,\rangle \;=\; \hat{\zeta}\,\delta^{ij}\,\delta(x_1 - x_2)$$

first order differential equation can be solved and then averaged

$$\langle\, \hat{\mathbf{p}}^2(x_2)\,\rangle \;=\; \langle\, \hat{\mathbf{p}}^2(x_1)\,\rangle\, \exp\left[-2\int_{x_1}^{x_2} \mathrm{d}y\, \hat{\eta}(y)\right] + 3\int_{x_1}^{x_2} \mathrm{d}z\, \hat{\zeta}(z)\, \exp\left[2\int_{x_2}^{z} \mathrm{d}y\, \hat{\eta}(y)\right]$$

numerical evaluation of the second moment



here the equilibrium value is $\langle \hat{\mathbf{p}}^2 \rangle_{\text{eq}} \equiv 3\hat{\zeta}/(2\hat{\eta}) \sim M/T$

approximate physics of the second moment

$$\langle\, \hat{\mathbf{p}}^2(x_2)\,\rangle \;\overset{\text{eq}}{\approx}\; \left[\, \langle\, \hat{\mathbf{p}}^2(x_1)\,\rangle - \frac{3\hat{\zeta}}{2\hat{\eta}} \right] e^{-2\hat{\eta}(x_2 - x_1)} \;+\; \frac{3\hat{\zeta}}{2\hat{\eta}}$$

$\hat{\eta}(x_2 - x_1) \ll 1$: $3\hat{\zeta}/(2\hat{\eta})$ cancels, so that non-equilibrium manifests itself by the system staying close to the old value

$\hat{\eta}(x_2 - x_1) \gg 1$: memory of initial conditions is lost, and the system moves towards $\langle \hat{\mathbf{p}}^2 \rangle_{\text{eq}} = 3\hat{\zeta}/(2\hat{\eta})$

summary: $x \sim \mathcal{O}(1) \Rightarrow$ kinetic decoupling starts when $\hat{\eta} < 1$

discretization of full langevin evolution

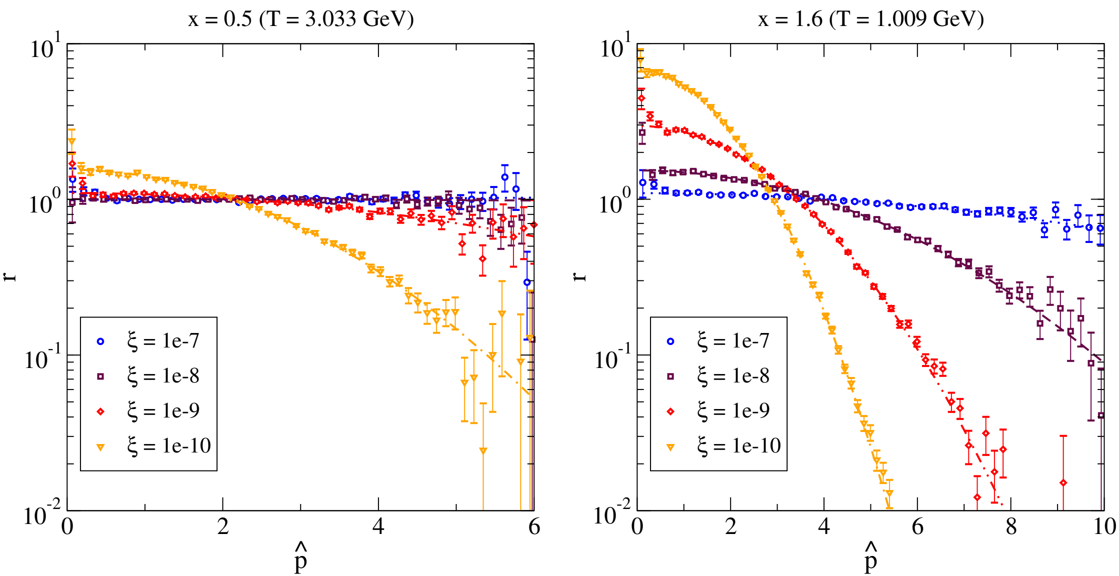
$$\hat{p}^{\,i}_{n+1} = \hat{p}^{\,i}_{n} - \hat{\eta}_n\,\hat{p}^{\,i}_{n}\mathrm{d}x + \hat{f}^{\,i}_{n}\sqrt{\mathrm{d}x}\ , \quad \langle\,\hat{f}^{\,i}_{n}\hat{f}^{\,j}_{m}\,\rangle = \hat{\zeta}_n\,\delta^{ij}\,\delta_{mn}$$

initial $\hat{p}^{\,i}$'s drawn from the equilibrium distribution at $T = 5$ GeV

histograms produced from $N = 10^5$ independent runs

errors from a jackknife analysis, with a block size of 10^3

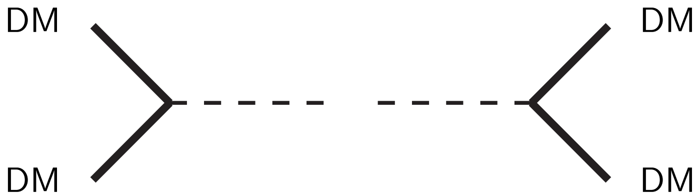
ratio of full and equilibrium distributions ($r \equiv \mathcal{P}/\mathcal{P}_{\rm eq}$)



fits ⇒ momentum distribution maintains a gaussian form even after the system falls out of equilibrium

implications for freeze-out

boltzmann equation for the number density



$$(\partial_t - H p_1 \partial_{p_1}) f_{\varphi_1} \approx$$

$$- \int_{\mathbf{p}_2, \mathbf{p}_h} \frac{\kappa^2 v^2 (2\pi)^4 \delta(\epsilon_{\varphi_1} + \epsilon_{\varphi_2} - \epsilon_h) \delta^{(3)}(\mathbf{p}_1 + \mathbf{p}_2 - \mathbf{p}_h)}{8 \epsilon_{\varphi_1} \epsilon_{\varphi_2} \epsilon_h}$$

$$\times (f_{\varphi_1} f_{\varphi_2} - \bar{f}_{\varphi_1} \bar{f}_{\varphi_2})$$

here the equilibrium form reads $\bar{f}_\varphi = \exp(-\epsilon_\varphi / T)$

after integration over momenta

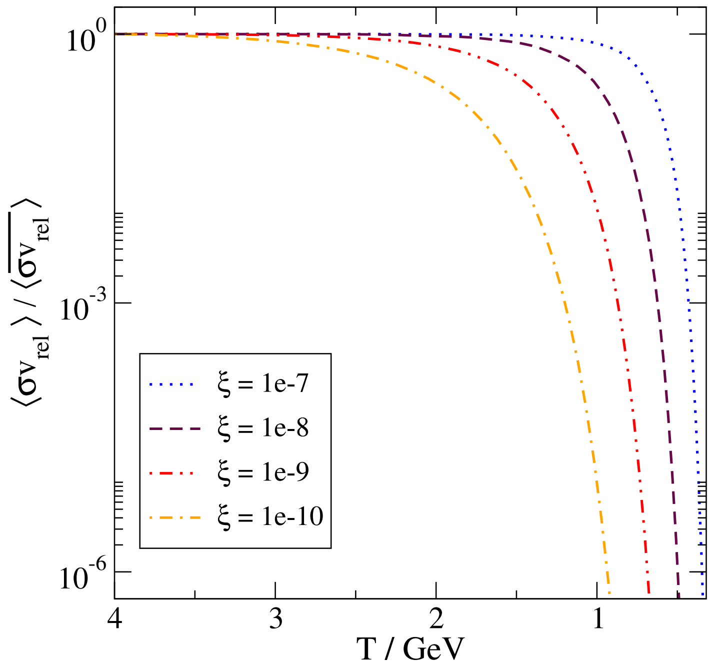
$$\partial_x Y_\varphi \approx -\frac{s}{3c_s^2 H}\left[\langle \sigma v_{\text{rel}}\rangle Y_\varphi^2 - \langle \overline{\sigma v}_{\text{rel}}\rangle \bar{Y}_\varphi^2\right]$$

where the dynamical variable is $Y \equiv \int_{\mathbf{p}} f_\varphi / s$

$\langle \sigma v_{\text{rel}}\rangle \equiv$ momentum average with respect to f_φ

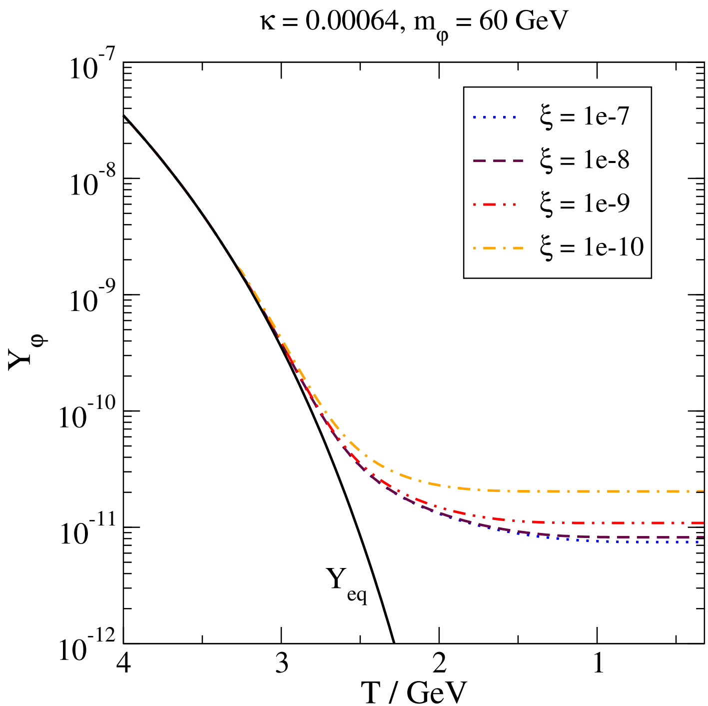
$\langle \overline{\sigma v}_{\text{rel}}\rangle \equiv$ momentum average with respect to $\bar{f}_\varphi$

cross section is reduced thanks to redshifted spectra



this is because of less weight in the high-momentum domain

therefore the yield is higher for more non-equilibrium



$\sim$40% increase of $Y_\varphi \Leftrightarrow \sim$20% increase of the coupling κ

summary

$\Rightarrow$ kinetic non-equilibrium may be important for precision studies

$\Rightarrow$ langevin simulations provide for an efficient framework for this

$\Rightarrow$ in the scalar singlet case, effects on the 40% level found

$\Rightarrow$ there are QCD effects but not as large as claimed

$\Rightarrow$ other models remain to be investigated