On gravitation as a medium property in Maxwell equations

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August 28, 2023

Axion electrodynamics and magnetohydrodynamics

$$L = \frac{c^4}{16\pi G}(R - 2\Lambda) - \frac{1}{2}\phi^{;c}\phi_{,c} - V(\phi) + L_{\rm m} - \frac{1}{4}F_{ab}F^{ab} - \frac{g_{\phi\gamma}}{4} f(\phi) F_{ab}F^{*ab} + \frac{1}{c}J^a A_a$$

Signature = (–, +, +, +)

Fluid

Field strength tensor (Heaviside unit)

Helical coupling Sikivie (1983)

Three-component system:

❖ **Gravity**: Einstein's equation

❖ **Fluid**: mass, energy, and momentum conservation equations

❖ **Scalar field**: Klein-Gordon equation

❖ **Electromagnetism**: Maxwell's equations

Four formulations of electrodynamics and MHD:

1. Spacetime covariant formulation
2. Spatially covariant ADM formulation
3. Fully-nonlinear and exact perturbation formulation (cosmological)

    Ignored transverse-tracefree perturbation

4. Weak gravity limit (cosmological)

JH, Noh, Exact formulations of relativistic electrodynamics and magnetohydrodynamics with helically coupled scalar field, 2211.03926

Axion electrodynamics

Inhomogeneous part: Homogeneous part:

❖ **Maxwell eqs:** $F^{ab}{}_{;b} = \frac{1}{c}J^a - g_{\phi\gamma} f_{,b} F^{*ab}$, $\quad \eta^{abcd}F_{bc,d} = 0.$

Dual Anti-symmetric tensor

$$F^{*ab}{}_{;b} = 0 \qquad F^*_{ab} \equiv \tfrac{1}{2}\eta_{abcd}F^{cd}$$

$$H^{ab}{}_{;b} = \frac{1}{c}J^a, \qquad H_{ab} \equiv F_{ab} + g_{\phi\gamma} f F^*_{ab}.$$

Electric and magnetic

❖ **EM fields:**

Time-like frame four-vector, normalizes with $u_a u^a \equiv -1$.

$$F_{ab} = u_a E_b - u_b E_a - \eta_{abcd} u^c B^d, \qquad E_a u^a \equiv 0 \equiv B_a u^a$$

$$H_{ab} = u_a D_b - u_b D_a - \eta_{abcd} u^c H^d.$$

❖ **Axion as an effective medium:**

Polarization

$$D_a \equiv E_a + P^{\mathrm{A}}_a, \qquad P^{\mathrm{A}}_a \equiv g_{\phi\gamma} f B_a,$$

$$H_a \equiv B_a - M^{\mathrm{A}}_a, \qquad M^{\mathrm{A}}_a \equiv g_{\phi\gamma} f E_a.$$

Magnetization

Sikivie (1983); Wilczek (1987)

Maxwell’s equations in curved spacetime

Maxwell’s equations in (spacetime) covariant form

Maxwell eqs: $F^{ab}{}_{;b} = \frac{1}{c} J^a - g_{\phi\gamma} f_{,b} F^{*ab}, \quad \eta^{abcd} F_{bc,d} = 0.$

Decomposition: $F_{ab} \equiv u_a E_b - u_b E_a - \eta_{abcd} u^c B^d \qquad J^a \equiv \varrho c u^a + j^a, \quad j_a u^a \equiv 0.$

Projection tensor   Rotation

Gauss:

$$E^a{}_{;b} h^b_a = -2\omega^a B_a + \varrho - g_{\phi\gamma} f_{,a} B^a$$

Acceleration

Ampère:

$$h^a_b \dot{\widetilde{E}}^b = \left(\eta^a{}_{bcd} u^d \omega^c + \sigma^a{}_b - \frac{2}{3}\delta^a_b \theta\right) E^b + \eta^{abcd} u_d \left(a_b B_c - B_{b;c}\right)$$

$\dot{\widetilde{E}}^a \equiv E^a{}_{;b} u^b$

$$- \frac{1}{c}\left[j^a + c g_{\phi\gamma} f_{,b}\left(u^b B^a - \eta^{abcd} u_c E_d\right)\right],$$

$$B^a{}_{;b} h^b_a = 2\omega^a E_a,$$

Shear tensor   Expansion

Faraday:

$$h^a_b \dot{\widetilde{B}}^b = \left(\eta^a{}_{bcd} u^d \omega^c + \sigma^a{}_b - \frac{2}{3}\delta^a_b \theta\right) B^b - \eta^{abcd} u_d \left(a_b E_c - E_{b;c}\right).$$

$$\Longleftarrow \quad u_a F^{ab}{}_{;b} = \dots, \quad h^a_c F^{cb}{}_{;b} = \dots, \quad u_a \eta^{abcd} F_{bc;d} = 0, \quad h^a_e \eta^{ebcd} F_{bc;d} = 0.$$

G.F.R. Ellis, Relativistic cosmology, in Cargese Lectures in Physics (1973); JH, Noh, 2211.03926

Maxwell's equations in (spatially covariant) ADM form

ADM metric:

$$g_{00} \equiv -N^2 + N^i N_i, \quad g_{0i} \equiv N_i, \quad g_{ij} \equiv \overline{h}_{ij}.$$

Lapse function

Shift vector

Intrinsic metric with $\overline{h}^{ik}\overline{h}_{jk} \equiv \delta^i_j$

Normal frame:
(Eulerian observer)

$$\widetilde{n}_i \equiv 0, \quad \widetilde{n}_0 = -N, \quad \widetilde{n}^i = -\frac{N^i}{N}, \quad \widetilde{n}^0 = \frac{1}{N}.$$

$$\widetilde{B}_i \equiv \overline{B}_i, \quad \widetilde{B}_0 = \overline{B}_i N^i, \quad \widetilde{B}^i = \overline{B}^i, \quad \widetilde{B}^0 = 0$$

Indices associated with $\overline{h}_{ij}$

Gauss:

$$\overline{\nabla} \cdot \overline{\mathbf{E}} = \varrho - g_{\phi\gamma} \overline{\mathbf{B}} \cdot \overline{\nabla} f,$$

Ampère:

$$\frac{1}{\sqrt{\overline{h}}}(\sqrt{\overline{h}}\overline{\mathbf{E}})_{,0} - \overline{\nabla} \times (N\overline{\mathbf{B}} - \mathbf{N} \times \overline{\mathbf{E}}) = \mathbf{N}\overline{\nabla} \cdot \overline{\mathbf{E}}$$

$$-\frac{1}{c} N \left\{ \overline{\mathbf{j}} + c g_{\phi\gamma} \left[\frac{1}{N} \overline{\mathbf{B}} \left(\partial_0 - \mathbf{N} \cdot \overline{\nabla} \right) f - \overline{\mathbf{E}} \times \overline{\nabla} f \right] \right\},$$

$$\frac{1}{\sqrt{\overline{h}}}(\sqrt{\overline{h}}\overline{B}^i)_{,i}$$

$$\overline{h} \equiv \det(\overline{h}_{ij})$$

$$\overline{\nabla} \cdot \overline{\mathbf{B}} = 0,$$

Faraday:

$$\frac{1}{\sqrt{\overline{h}}}(\sqrt{\overline{h}}\overline{\mathbf{B}})_{,0} + \overline{\nabla} \times (N\overline{\mathbf{E}} + \mathbf{N} \times \overline{\mathbf{B}}) = 0.$$

Textbooks on Numerical Relativity; JH, Noh, [2211.03926](2211.03926)

Maxwell's equations in FNLE form

Scale factor

Index associated with δ_{ij}

FNLE metric: $\tilde{g}_{00} = -a^2 (1 + 2\alpha), \quad \tilde{g}_{0i} = -a\chi_i, \quad \tilde{g}_{ij} = a^2 (1 + 2\varphi)\,\delta_{ij}$

Ignoring the TT part but without imposing temporal gauge condition yet

Fully nonlinear and exact perturbation

Index associated with $\overline{h}_{ij}$

Index associated with δ_{ij}

$$\overline{B}_i \equiv a B_i, \quad \overline{B}^i = \frac{B^i}{a(1+2\varphi)}$$

Gauss:

$$\frac{\nabla \cdot (\sqrt{1+2\varphi}\mathbf{E})}{a(1+2\varphi)^{3/2}} = \varrho - g_{\phi\gamma} \frac{\mathbf{B} \cdot \nabla f}{a(1+2\varphi)},$$

Ampère:

$$\frac{1}{c}\frac{\partial}{\partial t}\left(a^2\sqrt{1+2\varphi}\mathbf{E}\right) - \nabla \times \left(a\mathcal{N}\mathbf{B} + \frac{1}{\sqrt{1+2\varphi}}\vec{\chi} \times \mathbf{E}\right) = -\vec{\chi}\frac{\nabla \cdot (\sqrt{1+2\varphi}\mathbf{E})}{1+2\varphi}$$
$$-\frac{1}{c}a^2\mathcal{N}\sqrt{1+2\varphi}\left\{\mathbf{j} + g_{\phi\gamma}\left[\frac{1}{\mathcal{N}}\mathbf{B}\left(\dot{f} + \frac{c\vec{\chi}\cdot\nabla f}{a^2(1+2\varphi)}\right) - \frac{c\mathbf{E}\times\nabla f}{a\sqrt{1+2\varphi}}\right]\right\},$$

$$\nabla \cdot \left(\sqrt{1+2\varphi}\mathbf{B}\right) = 0,$$

Faraday:

$$\frac{1}{c}\frac{\partial}{\partial t}\left(a^2\sqrt{1+2\varphi}\mathbf{B}\right) + \nabla \times \left(a\mathcal{N}\mathbf{E} - \frac{1}{\sqrt{1+2\varphi}}\vec{\chi} \times \mathbf{B}\right) = 0.$$

$$\mathcal{N} \equiv \sqrt{1 + 2\alpha + \frac{\chi^k\chi_k}{a^2(1+2\varphi)}} = \frac{1}{a}N$$

JH, Noh, 2211.03926

Maxwell’s equations in the normal frame

Normal frame (Eulerian observer):

❖ **Maxwell's equations:**

$$(\sqrt{\overline{h}}\overline{h}^{ij}E_j)_{,i} = \sqrt{\overline{h}}\varrho,$$

$$(\sqrt{\overline{h}}\overline{h}^{ij}E_j)_{,0} - \eta^{ijk}\nabla_j(NB_k - \eta_{k\ell m}\sqrt{\overline{h}}N^\ell\overline{h}^{mn}E_n) = \sqrt{\overline{h}}N^i\varrho - \frac{1}{c}N\sqrt{\overline{h}}\overline{h}^{ij}j_j,$$

$$(\sqrt{\overline{h}}\overline{h}^{ij}B_j)_{,i} = 0,$$

$$(\sqrt{\overline{h}}\overline{h}^{ij}B_j)_{,0} + \eta^{ijk}\nabla_j(NE_k + \eta_{k\ell m}\sqrt{\overline{h}}N^\ell\overline{h}^{mn}B_n) = 0.$$

❖ **Effective polarizations and magnetizations:**

$$(E^i + P^i_{\mathrm{E}})_{,i} = \sqrt{\overline{h}}\varrho,$$

$$(E^i + P^i_{\mathrm{E}})_{,0} - \eta^{ijk}\nabla_j(B_k - M^{\mathrm{E}}_k) = \sqrt{\overline{h}}N^i\varrho - \frac{1}{c}N\sqrt{\overline{h}}\overline{h}^{ij}j_j,$$

$$(B^i + P^i_{\mathrm{B}})_{,i} = 0,$$

$$(B^i + P^i_{\mathrm{B}})_{,0} + \eta^{ijk}\nabla_j(E_k - M^{\mathrm{B}}_k) = 0.$$

$$P^i_{\mathrm{E}} \equiv \sqrt{\overline{h}}\overline{h}^{ij}E_j - E^i,$$

$$M^{\mathrm{E}}_i \equiv (1-N)B_i + \eta_{ijk}\sqrt{\overline{h}}N^j\overline{h}^{k\ell}E_\ell,$$

$$P^i_{\mathrm{B}} \equiv \sqrt{\overline{h}}\overline{h}^{ij}B_j - B^i,$$

$$M^{\mathrm{B}}_i \equiv (1-N)E_i - \eta_{ijk}\sqrt{\overline{h}}N^j\overline{h}^{k\ell}B_\ell.$$

Weak gravity and gravitational waves:

- **General linear perturbation in metric:**

Indices associated with η_{ab}

$$g_{ab} \equiv \eta_{ab} + h_{ab}, \quad g^{ab} = \eta^{ab} - h^{ab},$$

$$\Gamma^a_{bc} = \frac{1}{2}(h^a_{b,c} + h^a_{c,b} - h_{bc}{}^{,a}),$$

$$R^a{}_{bcd} = \frac{1}{2}(h^a_{d,bc} + h_{bc}{}^{,a}{}_d - h^a_{c,bd} - h_{bd}{}^{,a}{}_c).$$

- **Using**

Indices associated with δ_{ij}

$$\eta_{0ijk} = -\sqrt{-g}\eta_{ijk}, \quad \eta^{0ijk} = \frac{1}{\sqrt{-g}}\eta^{ijk},$$

$$g \equiv \det(g_{ab}) = -(1 + h^0_0 + h^i_i).$$

we have

$$N = 1 - \frac{1}{2}h_{00}, \quad N_i = h_{0i}, \quad N^i = h^i_0,$$

$$\overline{h}_{ij} = \delta_{ij} + h_{ij}, \quad \overline{h}^{ij} = \delta^{ij} - h^{ij}, \quad \overline{h} = 1 + h^i_i.$$

Maxwell's equations in weak gravity

❖ **Normal frame, to linear order:**

$$(E^i + P^i_{\rm E})_{,i} = \varrho \left(1 + \frac{1}{2} h^i_i \right),$$

$$(E^i + P^i_{\rm E})_{,0} - \eta^{ijk} \nabla_j (B_k - M^{\rm E}_k) = -\frac{1}{c} \left(1 + \frac{1}{2} h^j_j + \frac{1}{2} h^0_0 \right) j^i + \frac{1}{c} h^{ij} j_j + \varrho h^i_0,$$

$$(B^i + P^i_{\rm B})_{,i} = 0,$$

$$(B^i + P^i_{\rm B})_{,0} + \eta^{ijk} \nabla_j (E_k - M^{\rm B}_k) = 0.$$

$$P^i_{\rm E} \equiv \frac{1}{2} h^j_j E^i - h^{ij} E_j, \quad M^i_{\rm E} \equiv \frac{1}{2} h_{00} B^i + \eta^{ijk} h_{0j} E_k,$$

$$P^i_{\rm B} \equiv \frac{1}{2} h^j_j B^i - h^{ij} B_j, \quad M^i_{\rm B} \equiv \frac{1}{2} h_{00} E^i - \eta^{ijk} h_{0j} B_k.$$

❖ **Polarizations and magnetizations in both homogeneous and inhomogeneous parts!**

❖ **We can still impose four gauge conditions: E.g., synchronous gauge is** $h_{00} = 0 = h_{0i}$.

❖ **For gravitational waves, TT gauge condition is** $h_{0a} = 0$ **and** $h^{ij}{}_{,j} = 0 = h^i{}_i$.

JH, Noh, Definition of electric and magnetic fields in curved spacetime, 2303.07562

Electrodynamics in curved spacetime

Maxwell's equations: $F^{ab}{}_{;b} = \frac{1}{c} J^a, \quad \eta^{abcd} F_{bc,d} = 0$ $\qquad F^*_{ab} \equiv \frac{1}{2}\eta_{abcd}F^{cd}$

Special relativity:
$$F_{ab} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & B_z & -B_y \\ E_y & -B_z & 0 & B_x \\ E_z & B_y & -B_x & 0 \end{pmatrix}$$

General relativity: $F_{ab} = u_a E_b - u_b E_a - \eta_{abcd} u^c B^d$ $\qquad E_a u^a \equiv 0 \equiv B_a u^a$

$J^a \equiv \varrho c u^a + j^a, \quad j_a u^a \equiv 0$ $\qquad E_a \equiv F_{ab} u^b \qquad B_a \equiv F^*_{ab} u^b$

Not modified in the homogeneous part!

Errors in:

1. GW haloscope proposal using an analogy with axion

2. Gravity-medium analogy (light path vs. material property)

3. Transformation optics for metamaterials (controlling permittivity and permeability)

4. Graviton-photon conversion in the external magnetic field

∵ 1-4 used special relativistic definition.

Gertsenshtein mechanism (1962) ~ Primakoff effect (1951)

General relativistic definition → the homogeneous part of Maxwell's equations is also modified for any four-vector (observer) → no medium interpretation possible!

JH, Noh, Definition of electric and magnetic fields in curved spacetime, 2303.07562

History

Die Grundgleichungen für die elektromagnetischen Vorgänge in bewegten Körpern.*)

Von

HERMANN MINKOWSKI.

*) Abgedruckt aus den Nachrichten der Kgl. Ges. d. Wiss. zu Göttingen, Math.-phys. Kl., Sitzung vom 21. Dezember 1907.

$$\boxed{H^{ab}{}_{,b} = \frac{1}{c} J^a, \quad \eta^{abcd} F_{bc,d} = 0.}$$

Minkowski background (original)

❖ Minkowski (1907):

$$F_{ab} \equiv \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & B_z & -B_y \\ E_y & -B_z & 0 & B_x \\ E_z & B_y & -B_x & 0 \end{pmatrix}, \quad H_{ab} \equiv \begin{pmatrix} 0 & -D_x & -D_y & -D_z \\ D_x & 0 & H_z & -H_y \\ D_y & -H_z & 0 & H_x \\ D_z & H_y & -H_x & 0 \end{pmatrix}.$$

$$J^a \equiv (\varrho c, j^i)$$

❖ Maxwell equations:

Inhomogeneous part: $H^{ab}{}_{,b} = \frac{1}{c} J^a \Longrightarrow \begin{cases} \nabla \cdot \mathbf{D} = \varrho, \\ \mathbf{D}_{,0} - \nabla \times \mathbf{H} = -\frac{1}{c}\mathbf{j}, \end{cases}$

Homogeneous part: $\eta^{abcd} F_{bc,d} = 0 \Longrightarrow \begin{cases} \nabla \cdot \mathbf{B} = 0, \\ \mathbf{B}_{,0} + \nabla \times \mathbf{E} = 0. \end{cases}$

Minkowski background (using four-vector)

❖ **Normal (Eulerian) frame**: $n_0 = -1, \quad n_i \equiv 0, \quad n^0 = 1, \quad n^i = 0.$

$$F_{ab} \equiv n_a E_b - n_b E_a - \eta_{abcd} n^c B^d, \quad H_{ab} \equiv n_a D_b - n_b D_a - \eta_{abcd} n^c H^d.$$

$$F_{0i} = -E_i, \quad F_{ij} = \eta_{ijk} B^k, \quad H_{0i} = -D_i, \quad H_{ij} = \eta_{ijk} H^k.$$

$$F_{ab} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & B_z & -B_y \\ E_y & -B_z & 0 & B_x \\ E_z & B_y & -B_x & 0 \end{pmatrix}, \quad H_{ab} = \begin{pmatrix} 0 & -D_x & -D_y & -D_z \\ D_x & 0 & H_z & -H_y \\ D_y & -H_z & 0 & H_x \\ D_z & H_y & -H_x & 0 \end{pmatrix}.$$

$$J^a \equiv \varrho c n^a + j^a = (\varrho c, j^i)$$

❖ **Comoving (Lagrangian) frame**: $u_0 = -\underbrace{\gamma}_{\text{Lorentz factor}}, \quad u_i \equiv \gamma \frac{v_i}{c}, \quad u^0 = \gamma, \quad u^i = \gamma \frac{v^i}{c}.$

$$F_{ab} \equiv u_a e_b - u_b e_a - \eta_{abcd} u^c b^d, \quad H_{ab} \equiv u_a d_b - u_b d_a - \eta_{abcd} u^c h^d.$$

$$J^a = \varrho c u^a + j^a = (\varrho c \gamma + \frac{v_i}{c} j^i, \varrho \gamma v^i + j^i).$$

Minkowski background

❖ **Constitutive relation** (**medium property**, imposed in **comoving (Lagrangian) frame**):

permittivity

permeability

$e_a u^a \equiv 0$

Evaluate in normal frame

$$\begin{aligned} d_i &\equiv \varepsilon e_i \\ b_i &\equiv \mu h_i \end{aligned} \quad \Longrightarrow \quad \begin{aligned} d_a &\equiv \varepsilon e_a \\ b_a &\equiv \mu h_a \end{aligned} \quad \Longrightarrow \quad \begin{aligned} H_{ab}u^b &\equiv \varepsilon F_{ab}u^b \\ F^*_{ab}u^b &\equiv \mu H^*_{ab}u^b \end{aligned}$$

$$\Longrightarrow \quad \begin{aligned} \mathbf{D} + \frac{\mathbf{v}}{c} \times \mathbf{H} &= \varepsilon\left(\mathbf{E} + \frac{\mathbf{v}}{c} \times \mathbf{B}\right), \\ \mathbf{B} - \frac{\mathbf{v}}{c} \times \mathbf{E} &= \mu\left(\mathbf{H} - \frac{\mathbf{v}}{c} \times \mathbf{D}\right). \end{aligned}$$

❖ **Non-relativistic limit:** Normal frame

$$\begin{aligned} \mathbf{D} &= \varepsilon\mathbf{E} + (\mu\varepsilon - 1)\frac{\mathbf{v}}{c} \times \mathbf{H}, \\ \mathbf{B} &= \mu\mathbf{H} - (\mu\varepsilon - 1)\frac{\mathbf{v}}{c} \times \mathbf{E}. \end{aligned}$$

Minkowski (1907); Landau & Lifshitz, *Electrodynamics of Continuous Media* (1960) Sec. 57.

Doc. 27

[p. 184] Plenary session of February 3, 1916

A New Formal Interpretation of Maxwell's Field Equations of Electrodynamics

by A. Einstein

$$\boxed{F^{ab}{}_{;b} = \frac{1}{c} J^a, \quad \eta^{abcd} F_{bc,d} = 0.}$$

Or, using $F^*_{ab} \equiv \frac{1}{2}\eta_{abcd}F^{cd}$, equivalently,

$$\boxed{-\frac{1}{2}\eta^{abcd} F^*_{bc,d} = \frac{1}{c} J^a, \quad F^{*ab}{}_{;b} = 0.}$$

English translation in *The Collected Papers of Albert Einstein. Vol. 06 The Berlin Years: Writings, 1914-1917* (1997) p. 132.

❖ **Special relativistic** F_{ab} :

$$F_{ab} \equiv \begin{pmatrix} 0 & -\hat{E}_x & -\hat{E}_y & -\hat{E}_z \\ \hat{E}_x & 0 & \hat{B}_z & -\hat{B}_y \\ \hat{E}_y & -\hat{B}_z & 0 & \hat{B}_x \\ \hat{E}_z & \hat{B}_y & -\hat{B}_x & 0 \end{pmatrix}$$

Index associated with δ_{ij}

$$\eta^{abcd} F_{bc,d} = 0 \quad \Longrightarrow \quad \hat{B}^i{}_{,i} = 0, \quad \dot{\hat{B}}^i + c\eta^{ijk} \nabla_j \hat{E}_k = 0.$$

❖ **Special relativistic** F^*_{ab} :

$$F^*_{ab} \equiv \begin{pmatrix} 0 & -\breve{B}_x & -\breve{B}_y & -\breve{B}_z \\ \breve{B}_x & 0 & -\breve{E}_z & \breve{E}_y \\ \breve{B}_y & \breve{E}_z & 0 & -\breve{E}_x \\ \breve{B}_z & -\breve{E}_y & \breve{E}_x & 0 \end{pmatrix}$$

$$-\frac{1}{2}\eta^{abcd} F^*_{bc,d} = \frac{1}{c} J^a \quad \Longrightarrow \quad \breve{E}^i{}_{,i} = \frac{1}{c}\sqrt{-g} J^0, \quad \breve{E}^i{}_{,0} - \eta^{ijk} \nabla_j \breve{B}_k = -\frac{1}{c}\sqrt{-g} J^i.$$

❖ Einstein noticed these exact coincidences with special relativity and suggested Maxwell equations in covariant forms.

❖ According to Einstein, two vectors $\hat{E}_i$ and $\hat{B}_i$ are “in pretty complex relationship” with $\breve{E}_i$ and $\breve{B}_i$ which is determined by $F^*_{ab} = \frac{1}{2}\eta_{abcd} g^{ce} g^{df} F_{ef}$

❖ The EM fields defined in these non-covariant manners are **NOT** the ones measurable by any observer.

❖ **Special relativistic** F_{ab} **:**

$$(\hat{E}^i + \hat{P}^i_{\rm E})_{,i} = \frac{1}{c}\sqrt{-g}J^0,$$

$$(\hat{E}^i + \hat{P}^i_{\rm E})_{,0} - \eta^{ijk}\nabla_j(\hat{B}_k - \hat{M}^{\rm E}_k) = -\frac{1}{c}\sqrt{-g}J^i,$$

Special relativistic forms noticed by Einstein

$$\begin{cases} \hat{B}^i{}_{,i} = 0, \\ \hat{B}^i{}_{,0} + \eta^{ijk}\nabla_j\hat{E}_k = 0. \end{cases}$$

❖ **Special relativistic** F^*_{ab} **:**

$$\begin{cases} \breve{E}^i{}_{,i} = \frac{1}{c}\sqrt{-g}J^0, \\ \breve{E}^i{}_{,0} - \eta^{ijk}\nabla_j\breve{B}_k = -\frac{1}{c}\sqrt{-g}J^i, \end{cases}$$

$$(\breve{B}^i + \breve{P}^i_{\rm B})_{,i} = 0,$$

$$(\breve{B}^i + \breve{P}^i_{\rm B})_{,0} + \eta^{ijk}\nabla_j(\breve{E}_k - \breve{M}^{\rm B}_k) = 0.$$

$$\breve{E}^i = \frac{\sqrt{h}}{N}\overline{h}^{ij}\left(\hat{E}_j - \eta_{jk\ell}N^k\hat{B}^\ell\right) \equiv \hat{E}^i + \hat{P}^i_{\rm E},$$

$$\breve{B}_i = \frac{N}{\sqrt{h}}\left[\left(1 - \frac{N^kN_k}{N^2}\right)\overline{h}_{ij} + \frac{N_iN_j}{N^2}\right]\hat{B}^j - \frac{\sqrt{h}}{N}\eta_{ijk}N^j\overline{h}^{k\ell}\hat{E}_\ell \equiv \hat{B}_i - \hat{M}^{\rm E}_i,$$

$$\hat{B}^i = \frac{\sqrt{h}}{N}\overline{h}^{ij}\left(\breve{B}_j + \eta_{jk\ell}N^k\breve{E}^\ell\right) \equiv \breve{B}^i + \breve{P}^i_{\rm B},$$

$$\hat{E}_i = \frac{N}{\sqrt{h}}\left[\left(1 - \frac{N^kN_k}{N^2}\right)\overline{h}_{ij} + \frac{N_iN_j}{N^2}\right]\breve{E}^j + \frac{\sqrt{h}}{N}\eta_{ijk}N^j\overline{h}^{k\ell}\breve{B}_\ell \equiv \breve{E}_i - \breve{M}^{\rm B}_i.$$

"pretty complex relationship"

❖ Arbitrary, and cannot identify the frame four-vectors.

∴ Cannot define charge and current densities.

❖ These are **NOT** EM fields measurable by any observer!

❖ Coincidences guided Einstein, but he clearly noticed the nature of two different definitions and did not suggest these as the EM fields!

Minkowski background

- One can freely read $\mathbf{E}$ and $\mathbf{B}$ from components of F_{ab}, F^*_{ab}, or in any other combination.

$$F_{ab} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & B_z & -B_y \\ E_y & -B_z & 0 & B_x \\ E_z & B_y & -B_x & 0 \end{pmatrix}$$

$$F^*_{ab} = \begin{pmatrix} 0 & -B_x & -B_y & -B_z \\ B_x & 0 & -E_z & E_y \\ B_y & E_z & 0 & -E_x \\ B_z & -E_y & E_x & 0 \end{pmatrix}$$

- But **NOT** in curved spacetime!
- Even the above definitions in Minkowski spacetime are already based on the normal frame.

Covariant decomposition

(Møller 1952, Lichnerowicz 1967, Ellis 1973)

❖ The EM fields and charge and current densities, defined using a *generic* time-like frame four-vector u_a

$$F_{ab} = u_a E_b - u_b E_a - \eta_{abcd} u^c B^d \qquad E_a u^a \equiv 0 \equiv B_a u^a$$

$$\underset{\text{Four current}}{J^a} \equiv \underset{\text{Charge density}}{\varrho} c u^a + \underset{\text{Current density}}{j^a}, \quad j_a u^a \equiv 0$$

❖ Fluid quantities are similarly defined using the four-vector

$$T_{ab} = \mu u_a u_b + p h_{ab} + q_a u_b + q_b u_a + \pi_{ab},$$

Projection tensor $h_{ab} \equiv g_{ab} + u_a u_b$

$$\underset{\text{Energy density}}{\mu} = T_{ab} u^a u^b, \quad \underset{\text{Pressure}}{p} = \frac{1}{3} T_{ab} h^{ab}, \quad \underset{\text{Flux vector}}{q_a} = -T_{cd} u^c h_a^d, \quad \underset{\text{Anisotropic stress tensor}}{\pi_{ab}} = T_{cd} h_a^c h_b^d - p h_{ab}.$$

➢ Comoving four-vector (Lagrangian observer): $u_a =$ fluid four-vector.

➢ Normal four-vector (Eulerian observer): n_a with $n_i \equiv 0$.

➢ Coordinate four-vector: $\bar{n}_a$ with $\bar{n}^i \equiv 0$.

Gravity as a medium?

Eddington (1920)
Whitehead (1922)
Weyl (1923)
Tamm (1924)
…
Møller (1952)
Plebanski (1960)
Landau-Lifshitz (1971)

Plebanski (1960)

❖ **Definitions (non-covariant):**

Special relativistic forms!

ad hoc!

$$\boxed{\begin{aligned} &F_{0i} \equiv -\hat{E}_i, \quad F_{ij} \equiv \eta_{ijk}\hat{B}^k; \\ &\sqrt{-g}F^{0i} \equiv \hat{D}^i, \quad \sqrt{-g}F^{ij} \equiv \eta^{ijk}\hat{H}_k. \end{aligned}} \Longrightarrow (F^*_{0i} \equiv -\hat{H}_i, \quad F^*_{ij} \equiv -\eta_{ijk}\hat{D}^k)$$

❖ **Maxwell's equations:**

$$H^{ab}{}_{,b} = \frac{1}{c}J^a \Longrightarrow \begin{cases} \hat{D}^i{}_{,i} = \frac{1}{c}\sqrt{-g}J^0, \\ \hat{D}^i{}_{,0} - \eta^{ijk}\nabla_j \hat{H}_k = -\frac{1}{c}\sqrt{-g}J^i, \end{cases}$$

$$\eta^{abcd}F_{bc,d} = 0 \Longrightarrow \begin{cases} \hat{B}^i{}_{,i} = 0, \\ \hat{B}^i{}_{,0} + \eta^{ijk}\nabla_j \hat{E}_k = 0. \end{cases}$$

❖ **Constitutive relation:** from $\sqrt{-g}F^{ab} = \sqrt{-g}g^{ac}g^{bd}F_{cd}$.

Permittivity tensor **Permeability tensor** **Electromagnetic mixing**

$$\hat{D}^i = \varepsilon^{ij}\hat{E}_j + \gamma^{ij}\hat{H}_j, \quad \hat{B}^i = \mu^{ij}\hat{H}_j - \gamma^{ij}\hat{E}_j,$$

$$\varepsilon^{ij} = \mu^{ij} = \frac{\sqrt{-g}}{-g_{00}}g^{ij}, \quad \gamma^{ij} = \eta^{ijk}\frac{g_{0k}}{-g_{00}}.$$

Widely used in Transformation optics and metamaterial engineering: see Leonhardt & Philbin, General relativity in electrical engineering (2006).

Plebanski (1960)

❖ **Weak gravity:**

$$g_{00} = -(1+2\alpha), \quad g_{0i} = 0, \quad g_{ij} = (1+2\varphi)\delta_{ij}.$$

$$\alpha \equiv \frac{\Phi}{c^2}, \quad \varphi \equiv -\frac{\Psi}{c^2}, \quad \Psi = \Phi.$$

❖ **Static, spherical symmetry:**

$$\varphi = -\alpha = \frac{GM}{rc^2}.$$

❖ **Constitutive relations (medium property):**

$$\varepsilon^{ij} = \mu^{ij} = (1+\varphi-\alpha)\delta^{ij} \equiv \varepsilon\delta^{ij} = \mu\delta^{ij}, \quad \nu^{ij} = 0.$$

❖ **Refractive index:**

Correct value

$$n = \frac{c}{v} = \sqrt{\varepsilon\mu} = 1+\varphi-\alpha = 1-2\alpha = 1+2\frac{GM}{rc^2}.$$

Møller (1952), Landau-Lifshitz (1967)

❖ **Definitions (non-covariant):**

Special relativistic forms, but in $\overline{\gamma}_{ij}$ space.
∴ metric dependent!

ad hoc!

$$F_{0i} \equiv -\overline{E}_i, \quad F_{ij} \equiv \overline{\eta}_{ijk}\overline{B}^k,$$
$$\sqrt{-g_{00}}F^{0i} \equiv \overline{D}^i, \quad \sqrt{-g_{00}}F^{ij} \equiv \overline{\eta}^{ijk}\overline{H}_k.$$

$$\Longrightarrow \quad (F^*_{0i} = -\overline{H}_i, \quad F^*_{ij} = -\overline{\eta}_{ijk}\overline{D}^k)$$

indices associated with $\overline{\gamma}_{ij}$

❖ **Maxwell's equations:**

$$H^{ab}{}_{,b} = \frac{1}{c}J^a \quad \Longrightarrow \quad \begin{cases} \frac{1}{\sqrt{\overline{\gamma}}}\left(\sqrt{\overline{\gamma}}\overline{D}^i\right)_{,i} = \frac{1}{c}\overline{N}J^0, \\ \frac{1}{\sqrt{\overline{\gamma}}}\left(\sqrt{\overline{\gamma}}\overline{D}^i\right)_{,0} - \overline{\eta}^{ijk}\nabla_j\overline{H}_k = -\frac{1}{c}\overline{N}J^i, \end{cases}$$

$$\eta^{abcd}F_{bc,d} = 0 \quad \Longrightarrow \quad \begin{cases} \frac{1}{\sqrt{\overline{\gamma}}}\left(\sqrt{\overline{\gamma}}\overline{B}^i\right)_{,i} = 0, \\ \frac{1}{\sqrt{\overline{\gamma}}}\left(\sqrt{\overline{\gamma}}\overline{B}^i\right)_{,0} + \overline{\eta}^{ijk}\nabla_j\overline{E}_k = 0. \end{cases}$$

Homogeneous parts modified!

❖ **Constitutive relation:**

$$\overline{D}^i = \overline{\varepsilon}^{ij}\overline{E}_j + \overline{\nu}^{ij}\overline{H}_j, \quad \overline{B}^i = \overline{\mu}^{ij}\overline{H}_j - \overline{\nu}^{ij}\overline{E}_j,$$
$$\overline{\varepsilon}^{ij} = \overline{\mu}^{ij} = \frac{\overline{\gamma}^{ij}}{\overline{N}}, \quad \overline{\nu}^{ij} = \overline{\eta}^{ijk}\frac{\overline{N}_k}{\overline{N}^2}.$$

❖ **LL remark: "the [gravity-medium] analogy (purely formal, of course)"**

Møller, *The Theory of Relativity* (1952) Sec.115; Landau & Lifshitz, *The Classical Theory of Fields* (1967) Fifth Russian Edition, Sec. 90.

Spatially covariant decompositions

❖ **ADM (Arnowitt-Deser-Misner):**

Index associated with $\overline{h}_{ij}$

$$g_{00} \equiv -N^2 + N^i N_i, \quad g_{0i} \equiv N_i, \quad g_{ij} \equiv \overline{h}_{ij},$$

$$g^{00} = -\frac{1}{N^2}, \quad g^{0i} = \frac{N^i}{N^2}, \quad g^{ij} = \overline{h}^{ij} - \frac{N^i N^j}{N^2}.$$

Inverse metric of $\overline{h}_{ij}$

❖ **Møller, Landau-Lifshitz:**

Index associated with $\overline{\gamma}_{ij}$

$$g_{00} \equiv -\overline{N}^2, \quad g_{0i} \equiv \overline{N}_i, \quad g_{ij} = \overline{\gamma}_{ij} - \frac{\overline{N}_i \overline{N}_j}{\overline{N}^2},$$

Inverse metric of $\overline{\gamma}_{ij}$

$$g^{00} = -\frac{1}{\overline{N}^2}\left(1 - \frac{\overline{N}^k \overline{N}_k}{\overline{N}^2}\right), \quad g^{0i} = \frac{\overline{N}^i}{\overline{N}^2}, \quad g^{ij} \equiv \overline{\gamma}^{ij}.$$

❖ **Metric comparison:**

$$ds^2 = -N^2 dx^0 dx^0 + \overline{h}_{ij}\left(dx^i + N^i dx^0\right)\left(dx^j + N^j dx^0\right),$$

$$ds^2 = -\overline{N}^2 \left(dx^0 - \frac{\overline{N}_i}{\overline{N}^2} dx^i\right)\left(dx^0 - \frac{\overline{N}_j}{\overline{N}^2} dx^j\right) + \overline{\gamma}_{ij} dx^i dx^j.$$

Møller (1952); Landau-Lifshitz (1967)

❖ **Definitions (non-covariant):**

Indices associated with $\overline{\gamma}_{ij}$

Metric dependence!

$$\boxed{\begin{aligned} &F_{0i} = -E_i, \quad F_{ij} = \sqrt{\overline{\gamma}}\eta_{ijk}\overline{\gamma}^{k\ell}B_\ell; \\ &\sqrt{-g}F^{0i} = \sqrt{\overline{\gamma}}\overline{\gamma}^{ij}D_j, \quad \sqrt{-g}F^{ij} = \eta^{ijk}H_k. \end{aligned}} \quad (F^*_{0i} = -H_i, \quad F^*_{ij} = -\sqrt{\overline{\gamma}}\eta_{ijk}\overline{\gamma}^{k\ell}D_\ell)$$

Indices associated with δ_{ij}

❖ **Maxwell's equations:**

$$\frac{1}{\sqrt{\overline{\gamma}}}\left(\sqrt{\overline{\gamma}}\overline{\gamma}^{ij}D_j\right)_{,i} = \frac{1}{c}\overline{N}J^0,$$

$$\frac{1}{\sqrt{\overline{\gamma}}}\left(\sqrt{\overline{\gamma}}\overline{\gamma}^{ij}D_j\right)_{,0} - \frac{1}{\sqrt{\overline{\gamma}}}\eta^{ijk}\nabla_j H_k = -\frac{1}{c}\overline{N}J^i,$$

$$\left.\begin{aligned} &\frac{1}{\sqrt{\overline{\gamma}}}\left(\sqrt{\overline{\gamma}}\overline{\gamma}^{ij}B_j\right)_{,i} = 0, \\ &\frac{1}{\sqrt{\overline{\gamma}}}\left(\sqrt{\overline{\gamma}}\overline{\gamma}^{ij}B_j\right)_{,0} + \frac{1}{\sqrt{\overline{\gamma}}}\eta^{ijk}\nabla_j E_k = 0. \end{aligned}\right\}$$

Homogeneous parts modified!

❖ **Constitutive relation:**

$$\varepsilon_i^j = \mu_i^j = \frac{1}{\sqrt{-g_{00}}}\delta_i^j, \quad \nu_i^j = -\eta_{ik\ell}\frac{\sqrt{-g}}{\sqrt{-g_{00}}}g^{0k}g^{j\ell}.$$

Differ from Plebanski's

Møller, *The Theory of Relativity* (1952) Sec.115; Landau & Lifshitz, *The Classical Theory of Fields* (1967) Sec. 90.

Møller (1952); Landau-Lifshitz (1967)

- **Weak gravity, static, spherical symmetry:**

Missing 2 factor

$$\mu = \varepsilon = 1 - \alpha, \quad \sqrt{\mu\varepsilon} = 1 - \alpha = 1 + \frac{GM}{rc^2}.$$

Medium property

- **Maxwell's equations:**

$$[(1+\varphi)\varepsilon E^i]_{,i} = 0,$$
$$[(1+\varphi)\varepsilon E^i]_{,0} = \eta^{ijk}\nabla_j(\mu^{-1}B_k),$$
$$[(1+\varphi)B^i]_{,i} = 0,$$
$$[(1+\varphi)B^i]_{,0} = -\eta^{ijk}\nabla_j E_k.$$

$$\Longrightarrow \quad B^i{}_{,00} = (1-2\varphi)\frac{1}{\mu\varepsilon}\Delta B^i.$$

- **Refractive index:**

Correct value recovered

$$n \equiv \frac{c}{v} = (1+\varphi)\sqrt{\mu\varepsilon} = 1 - \alpha + \varphi = 1 + 2\frac{GM}{rc^2}.$$

Additional effect of gravity

Generic observer:

Indices associated with $\overline{h}_{ij}$

❖ **Generic four-vector:**

$$u_i \equiv \gamma V_i, \quad u_0 = \gamma\left(N_i V^i - N\right), \quad u^i = \gamma\left(V^i - \frac{1}{N}N^i\right), \quad u^0 = \frac{1}{N}\gamma. \qquad \gamma \equiv \frac{1}{\sqrt{1 - V^k V_k}}.$$

Index associated with δ_{ij}

$$\widetilde{B}_i \equiv B_i; \quad \widetilde{B}_0 = (N^i - NV^i)B_i, \quad \widetilde{B}^i = \overline{h}^{ij}B_j - \frac{N^i}{N}V^j B_j, \quad \widetilde{B}^0 = \frac{1}{N}V^i B_i.$$

❖ **Field strength tensor:**

Indices associated with $\overline{h}_{ij}$

$$F_{0i} = -\gamma(N - N_j V^j)E_i - \gamma V_i(N^j - NV^j)E_j + \gamma\overline{\eta}_{ijk}\left[(NV^j - N^j)\overline{h}^{k\ell}B_\ell - V^j N^k V^\ell B_\ell\right],$$
$$F_{ij} = \gamma(V_i E_j - V_j E_i) + \gamma\overline{\eta}_{ijk}(\overline{h}^{k\ell}B_\ell - V^k V^\ell B_\ell).$$

❖ **To linear order:**

$$F_{0i} = -NE_i + \overline{\eta}_{ijk}(NV^j - N^j)\overline{h}^{k\ell}B_\ell,$$

Six relations

$$F_{ij} = V_i E_j - V_j E_i + \overline{\eta}_{ijk}\overline{h}^{k\ell}B_\ell.$$

Only three degrees of freedom

❖ F_{ab} **without metric is not possible for any** u_a**.**

❖ **Gauge fixing does not help:** E.g., In the normal frame ($V_i = 0$) and synchronous gauge ($N = 1$, $N_i = 0$), we have $n_a = (-1, 0, 0, 0)$, but metric dependence remain in $\overline{\eta}_{ijk}$ and $\overline{B}^k$.

❖ **Therefore, gravity modifies the homogeneous part inevitably.**

Two wrong arguments-1:

Gravitational wave detection using axion haloscope by Berlin et al (2022), **confused the implication of**

- **The homogeneous part is topological; thus, no metric is involved:**

$$dF = 0$$

- **This does not imply that no metric is involved in the relation between F_{ab} and the EM fields.**
- **The inhomogeneous part is topological as well:**

$$d\,{}^*F = 4\pi\,{}^*J$$

- "Remarkably, neither equation makes any reference whatsoever to *metric*. [T]he concepts of form and exterior derivative are metric-free. Metric made an appearance only in one place, in the concept of duality ("perpendicularity") that carried attention from ***F*** to the dual structure ****F***." (MTW 1973)

$$F^*_{ab} \equiv \frac{1}{2}\eta_{abcd}F^{cd}$$

Metric involved

- **Metric is involved in relating the two parts and in the EM field decomposition.**

Misner, Thorne & Wheeler, *Gravitation* (1973) p. 114.

Two wrong arguments-2:

Domcke et al (2022)
confused the implication of

- **Four-potential:** $F_{ab} \equiv A_{b;a} - A_{a;b} = A_{b,a} - A_{a,b}$. **∴ no metric dependence!**
  Thus, $\eta^{abcd} F_{bc,d} = 0$ **is identically satisfied without metric influence.**
- **Minkowski spacetime (normal frame):** $n_a = (-1, 0, 0, 0), \quad n^a = (1, 0, 0, 0).$

$$F_{0i} \equiv -E_i, \quad F_{ij} \equiv \eta_{ijk} B^k. \quad \Longrightarrow \quad E_i = A_{0,i} - A_{i,0}, \quad B_i = \eta_{ijk} \nabla^j A^k.$$
$$(\mathbf{E} = \nabla A_0 - \mathbf{A}_{,0}, \quad \mathbf{B} = \nabla \times \mathbf{A})$$

- **Curved spacetime (normal frame):**

$$F_{0i} = -N E_i - \sqrt{\bar{h}} \eta_{ijk} N^j \bar{h}^{k\ell} B_\ell, \qquad E_i = \frac{1}{N} \left[\partial_i A_0 - A_{i,0} - (\partial_i A_j - \partial_j A_i) N^j \right],$$
$$F_{ij} = \sqrt{\bar{h}} \eta_{ijk} \bar{h}^{k\ell} B_\ell. \quad \Longrightarrow \quad B_i = \frac{1}{\sqrt{\bar{h}}} \bar{h}_{ij} \eta^{jk\ell} \partial_k A_\ell.$$

- **Using** $F_{ab} = A_{b,a} - A_{a,b}$, F_{ab} **is independent of metric, thus:**

$$F_{0i} \equiv -\hat{E}_i, \quad F_{ij} \equiv \eta_{ijk} \hat{B}^k. \quad \Longrightarrow \quad \hat{E}_i = A_{0,i} - A_{i,0}, \quad \hat{B}_i = \eta_{ijk} \nabla^j A^k.$$

- **∴ No metric involved in its relation to the four-potential does not imply that no metric is involved in the relation between** F_{ab} **and the EM fields.**
- **Metric appears in the EM field decomposition.**

Conclusion

- Definition of electric and magnetic (EM) fields should be made in a covariant way using a timelike frame four-vector as the observer's four-velocity.
- The normal four-vector is normal to the hypersurface and is the four-velocity of an Eulerian observer instantaneously at rest in the chosen time slice. To *any* observer, gravity appears as the effective polarizations and magnetizations in *both* the homogeneous and inhomogeneous parts of Maxwell's equations.
- Therefore, contrary to common wisdom, gravity *cannot* be regarded as an effective medium.
- There are popular misconceptions in the literature concerning F_{ab}.
- For optical properties in curved spacetime, gravitational wave detections, and photon-graviton interactions in the presence of a background magnetic field, Maxwell's equations in the normal frame should be analyzed directly.