Fine tuning as Quantum Phase Transition point

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Based on collaboration with H. Kawai, K. Oda, K. Yagyu:
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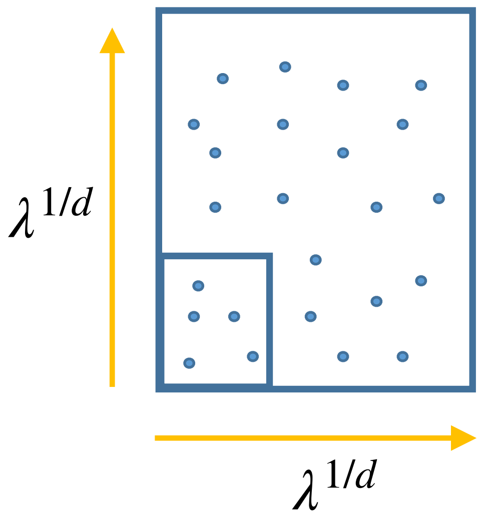
Extensive and intensive variables
and
Fine-tuning point of view

extensive and intensive variables

When a total system consists of λ identical subsystems

extensive variable $\mathcal{O} \propto \lambda^1$

intensive variable $\mathcal{O} \propto \lambda^0$



Example molecules in a box

extensive var.

V: volume

N: # of molecules

E: total energy

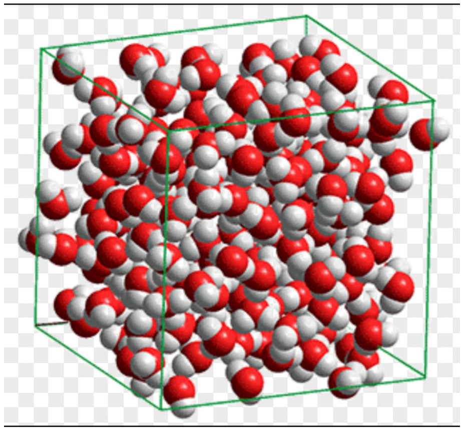
S: total entropy

intensive var.

p: pressure

μ: chemical pot.

T: temperature



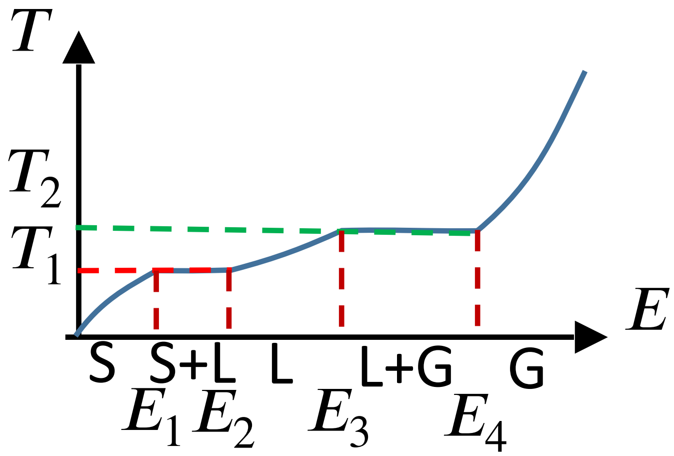
Tuning of intensive parameters without fine tuning

Micro-canonical picture: Extensive variables (E,V,N) are controlled.

$\Downarrow$

Corresponding intensive variables are given as functions of extensive variables

Example : Molecules in a box (with fixed V and N)



T is typically an increasing function of E

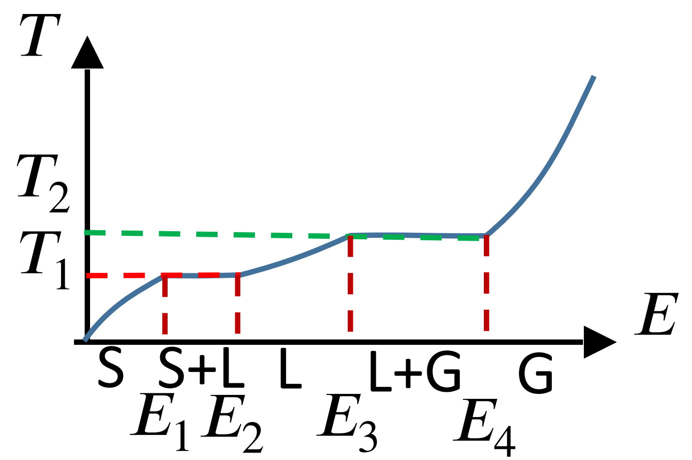
$\Leftarrow$ Phase transitions can happen

Tuning of intensive parameters without fine tuning

During a phase transition

the mapping $E \mapsto T$ is not $1:1$.

I.e. $T(E) =$ constant during a phase transition



$\Longrightarrow$ (First-order) Phase transition point spans a finite region of E

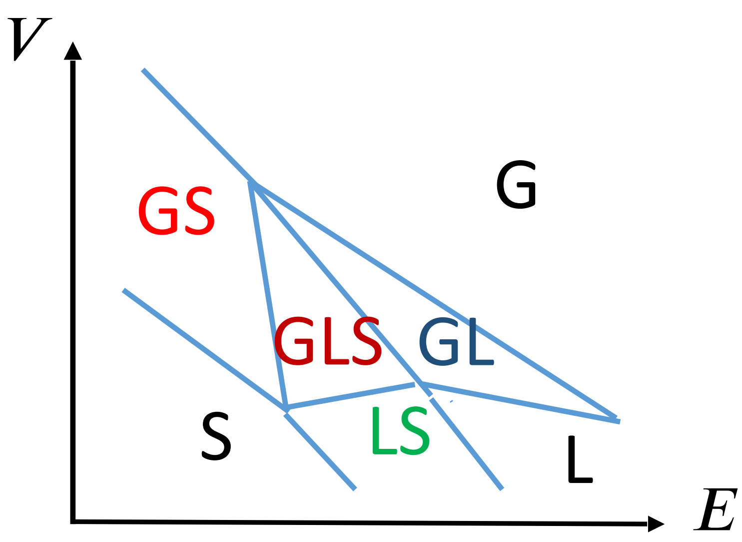
$\ast$ Fine-tuning viewpoint: When we control E, phase transition point is most likely realized with a finite probability $P \propto \frac{\Delta E}{E}$

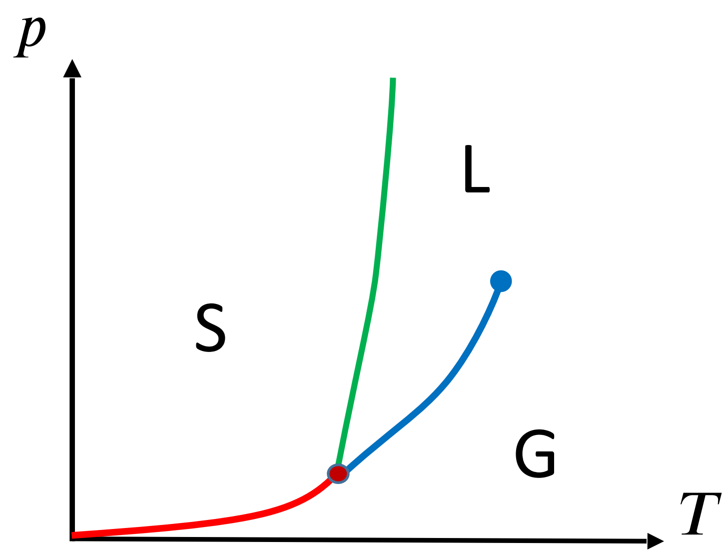
$\Longrightarrow$ T is automatically tuned to the critical value T_*.

In the canonical picture, this is usually seen as "fine tuning" because T is a controllable free parameter

More general case : Two control parameters V, E

molecules in a box with fixed N





Left: Phse transition points correspond to finte regions

Right: Phase transition points correspond to lines (fine-tuned)

＊ Even the triple point can be realized with a finite probability when the extensive parameters are controlled.

Question: How about in Quantum Field theory ?

We usually start from canonical picture

$$Z(m^2, \lambda, \cdots) = \int \mathscr{D}\phi \exp\left(iS_{\text{Kin}} - i\frac{m^2}{2}\int d^d x \phi(x)^2 - i\frac{\lambda}{4!}\int d^d x \phi(x)^4 + \cdots \right)$$

The coupling constants $(m^2, \lambda, \cdots)$ are intensive free parameters

$\Longrightarrow$ No reason to pic up specific values = Fine-tuning problems !

But, why should we start form this picture ?

Fine tuning of couplings might be automatically (naturally) realized if we start from micro-canonical QFT

$\Longrightarrow$ I will explain how to construct it and show that it is equivalent to the ordinary QFT with fine-tuned couplings in some cases

Quadratic divergence problem and strong CP problem can be naturally explained

Micro-canonical QFT
and Fine-tuning mechanism

Ordinary Micro-canonical ensemble

- Count the number of states (configurations) with a fixed energy

$$\Omega(E) = \sum_n \delta(E_n - E) = \mathrm{Tr}\left(\delta(\hat{H} - E)\right)$$

- More generally, when there exist other conserved charges $\{\hat{Q}_i\}$

$$\Omega(E, \{N_i\}) = \sum_n \delta(E_n - E)\left(\prod_i \delta(Q_{i,n} - N_i)\right) = \mathrm{Tr}\left(\delta(\hat{H} - E)\prod_i \delta(\hat{Q}_i - N_i)\right) .$$

- In the thermodynamic limit, temperature $T = \beta^{-1}$ is determined by the saddle point

$$\Omega(E) = \sum_n \delta(E_n - E) = \int_{-i\infty}^{i\infty} \frac{d\beta}{2\pi} \sum_n e^{-\beta(E_n - E)} = \int_{-i\infty}^{i\infty} \frac{d\beta}{2\pi} e^{-\beta(F(\beta) - E)} \sim e^{-\beta_*(F(\beta_*) - E)}$$

where β_* is a solution of $\quad \dfrac{\partial(\beta F(\beta))}{\partial\beta} = E$

Generalized Micro-canonical ensemble

K.K, H. Kawai, K. Yagyu, K.Oda (`23)
Bennett, Froggatt and Nielsen ('96)

- Not only Hamiltonian, there exist many other Hermitian operators

$$\mathcal{O}_i(x) = \phi(x)^n \ , (\partial\phi(x))^n \ , \overline{\psi}(x)\psi(x) \ , \phi(x)\overline{\psi}(x)\psi(x) \ , \cdots$$

- We can similarly define ``micro-canonical" ensemble

$$\Omega(\{A_i\}) = \int \mathcal{D}\phi \prod_i \delta\left(\int d^d x \mathcal{O}_i(x) - A_i \right)$$

Generalized micro-canonical ensemble

- A_i corresponds to energy E in the ordinary micro-canonical case.
- An intensive parameter g_i corresponding to $\mathcal{O}_i$ is nothing but a coupling constant

$$T \leftrightarrow g$$

$$E \leftrightarrow \int d^d x \mathcal{O}(x)$$

How is g fixed in the large volume limit ($V \to \infty$) ?

General argument for fine tuning of coupling constants

- For simplicity, we focus on one operator $S = \int d^d x \mathcal{O}(x)$

$$\Omega[A] = \int \mathscr{D}\phi \delta(S - A)$$

$\Leftarrow$ Express the delta function by Fourier modes

$$= \int_{-\infty}^{\infty} \frac{dg}{2\pi} \times \int \mathscr{D}\phi e^{ig(S-A)}$$

Ordinary QFT with coupling constant g.

- In statistical mechanics, there usually exists a saddle point

$\Longrightarrow$ corresponds to the fixing of intensive parameters $(T, p, \mu, \cdots)$

Question: Same is true for QFT ?

Besides, even if the answer is yes, what is the fixed value ?

Does it solve the fine-tuning problems ?

As an example, let us consider one of the notorious problems in QFT: Quadratic divergence problem

$$m_{\text{bare}}^2 + c\Lambda^2 = \mathcal{O}(100)\ \text{GeV}$$

We treat the scalar mass term in micro-canonical way i.e.

$$\Omega[A] = \int \mathscr{D}\phi e^{i\int d^d x\left(-\frac{1}{2}(\partial\phi)^2 - \frac{\lambda_B}{4}\phi^4\right)} \times \delta\left(\frac{1}{2}\int d^d x \phi(x)^2 - A\right)$$

$$= \int_{-\infty}^{\infty} \frac{dm_B^2}{2\pi} e^{im_B^2 A} \int \mathscr{D}\phi e^{i\int d^d x\left(\frac{1}{2}(\partial\phi)^2 - \frac{m_B^2}{2}\phi^2 - \frac{\lambda_B}{4!}\phi^4\right)}$$

Ordinary QFT

$$:= \int_{-\infty}^{\infty} \frac{dm_B^2}{2\pi} e^{im_B^2 A - iV_d f(m_B^2)}$$

$f(m_B^2)$ = free (vacuum) energy

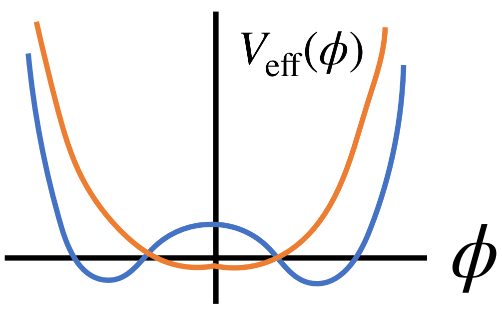
Where is the dominant point in this integration ?

$$\Omega(A) = \int_{-\infty}^{\infty} \frac{dm_B^2}{2\pi} e^{im_B^2 A - iV_d f(m_B^2)}$$

- We can calculate $f(m_B^2)$ perturbatively in the ordinary QFT
- But its detailed form is not important to determine the dominant point $m_B^2 = m_{*B}^2$

$\Longrightarrow$ Here I give very intuitive argument to find such a dominant point

Step 1: The main contribution to $f(m_B^2)$ is vacuum energy, which is determined by the effective potential $V_{\text{eff}}(\phi)$



Step 2: When a system has $\mathbb{Z}_2$ symmetry, $V_{\text{eff}}(\phi)$ always behaves as

$$V_{\text{eff}}(\phi) = \Lambda(m^2) + \frac{m^2}{2}\phi^2 + \frac{\lambda}{4}\phi^4 + \cdots \quad \text{where} \quad \begin{cases} m^2 = m_B^2 + \delta_{\text{loop}} m_B^2 \\ \lambda = \lambda_B + \delta_{\text{loop}} \lambda \end{cases}$$

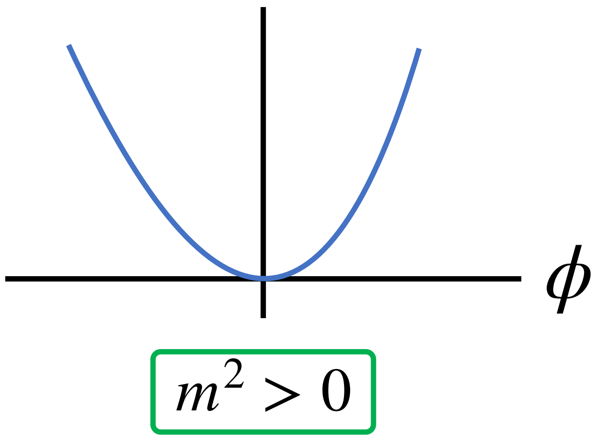
are Renormalized couplings (Not fixed yet !)

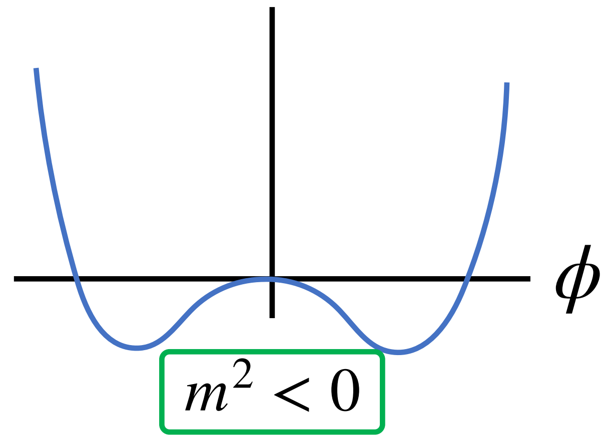
Contain UV divergences

$$V_{\text{eff}}(\phi) = \Lambda(m^2) + \frac{m^2}{2}\phi^2 + \frac{\lambda}{4}\phi^4 + \cdots$$

To guarantee the stability, we simply assume $\lambda > 0$

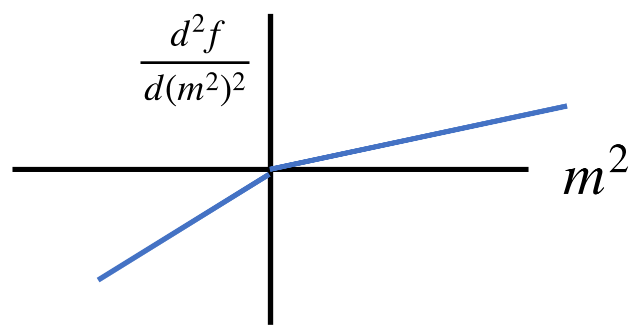
Step 3: Then, typical behavior of $V_{\text{eff}}(\phi)$ is





$$\therefore \quad f(m^2) = \begin{cases} \Lambda(m^2) & m^2 \geq 0 \\ \Lambda(m^2) - c \times (m^2)^2 & m^2 < 0 \end{cases}$$

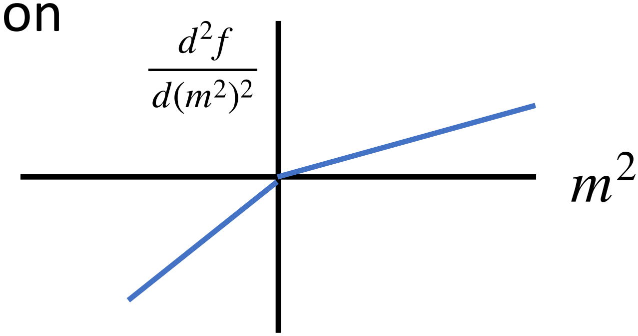
In particular, the second derivative of $f(m^2)$ is discontinuous at $m^2 = 0$



Step 4: Recall the micro-canonical partition function

$$\Omega(A) = \int_{-\infty}^{\infty} \frac{dm_B^2}{2\pi} e^{im_B^2 A - iV_d f(m_B^2)}$$

Second derivative is discontinuous



Then use the following Mathematical formula

Theorem: When (1) the second derivative of $f(x)$ is discontinuous at $x = x_0$ and (2) $f(x), f'(x)$ is monotonic and smooth for both $(-\infty, x_0]$ and $[x_0, \infty)$

$$e^{iVf(x)} \sim \frac{c}{V^2}\left[f''(x_0+0)^{-1} - f''(x_0-0)^{-1}\right] \times \delta(x - x_0) \qquad \text{for } V \to \infty$$

∵ By multiplying a test function $\varphi(x)$ with a finite support and performing the integration

$$\left(\int_{-\infty}^{x_0} dx + \int_{x_0}^{\infty} dx\right) e^{iVf(x)}\varphi(x) = \left(\int_{-\infty}^{f(x_0)} df + \int_{f(x_0)}^{\infty} df\right) e^{iVf} \frac{dx}{df}\varphi(x(f))$$

Use the monotonicity

↓ partial integration

$$= \frac{1}{iV}\left(\frac{dx}{df}\varphi\bigg|_{x_0+0} - \frac{dx}{df}\varphi\bigg|_{x_0}\right) + \frac{1}{iV}\left(\int_{-\infty}^{f(x_0)} df + \int_{f(x_0)}^{\infty} df\right) e^{iVf}\frac{d}{df}\left(\frac{dx}{df}\varphi(x(f))\right)$$

$= 0$ because first derivative is continuous

By repeating the same calculation

$$= \frac{c}{(iV)^2}\left(\left(\frac{d^2f}{dx^2}\right)^{-1}\varphi\bigg|_{x_0+0} - \left(\frac{d^2f}{dx^2}\right)^{-1}\varphi\bigg|_{x_0}\right) + \mathcal{O}(V^{-3})$$

$$\therefore\ e^{iVf(x)} \sim \frac{1}{V^2}\left[f''(x_0+0)^{-1} - f''(x_0-0)^{-1}\right] \times \delta(x - x_0) \qquad \text{for } V \to \infty \text{ //}$$

$\Longrightarrow$ Going back to the micro-canonical scalar theory, we finally get

$$\Omega(A) = \int_{-\infty}^{\infty} \frac{dm_B^2}{2\pi} e^{im_B^2 A - iV_d f(m_B^2)} \sim \mathcal{N} e^{-iV_d f(m^2=0)} = \mathcal{N} \times Z(m^2 = 0) \quad \text{for } V_d \to \infty$$

$\therefore$ Micro-canonical ensemble is equivalent to canonical ensemble

with $m^2 = 0$ $\Longrightarrow$ Automatic tuning (renormalization) !

Several remarks

(1) $m^2 = 0$ corresponds to the second-order phase transition point

(2) In this sense, the tuning mechanism is completely parallel to statistical mechanics (though the order of transition is different)

(3) But, the way of tuning is more general (due to Lorentzian signature)

- Statistical mechanics $\Rightarrow$ saddle point $T = T_*(E)$
- QFT $\Rightarrow$ Saddle point or quantum phase-transition point $g = g_*(A)$

e.g. Strong CP problem, $\theta = 0$, can be (trivially) explained as saddle point case

New viewpoint of fine tuning

In micro-canonical QFT, the coupling constants are automatically adjusted either to minimize the vacuum energy (saddle point) or to the critical points of quantum phase transition.

Summary

- We have discussed micro-canonical formulation of QFT
- In this picture, coupling constants (intensive parameters) are fixed at either (i) saddle point or (ii) quantum phase-transition point

In other words,

Fine tuning = a strongly dominant point in micro-canonical ensemble

- Tuning of mass parameter $m^2 = 0$ is a good example of phase-transition case (second order phase transition)
- More generally, there can be various (quantum) phase-transition points in coupling space

e.g. no $\mathbb{Z}_2$ invariant case, gauge coupling, Yukawa coupling, etc…

$\Longrightarrow$ They can be also fixed in micro-canonical picture (work in progress)

Thank you for your attention !

Backup slides

θ vacuum and Strong CP problem

- Let us treat the θ term in micro-canonical way

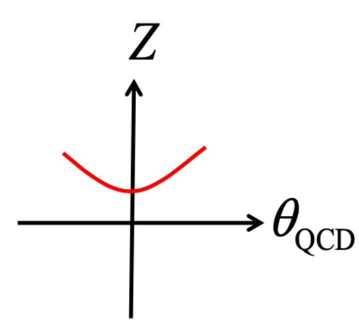
$$\Omega(A) = \int_0^{2\pi} d\theta e^{iV_d U(\theta)} \,. \qquad U(\theta) = \text{effective potential of } \theta$$

- $\tilde{G}^{\mu\nu}G_{\mu\nu}$ is Parity odd. But, action is invariant under $x^i \to -x^i$, $\theta \to -\theta$

$\therefore$ Physical observables such as vacuum energy, hadron mass, life time should be invariant under $\theta \to -\theta$ $\quad \therefore U(\theta) = U(-\theta)$

$\Rightarrow$ $\theta = 0$ must be local extremum

$\Rightarrow$ Saddle point case !

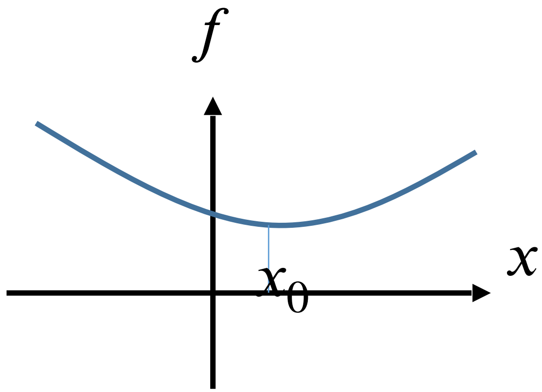


$$\Omega(A) = \int_0^{2\pi} d\theta e^{-iV_d U(\theta)} \sim e^{-iV_d U(\theta=0)}$$

Formulas on $e^{i\,Vf(x)}$ for large V

$e^{iVf(x)}$ for large V $f(x)$: real function

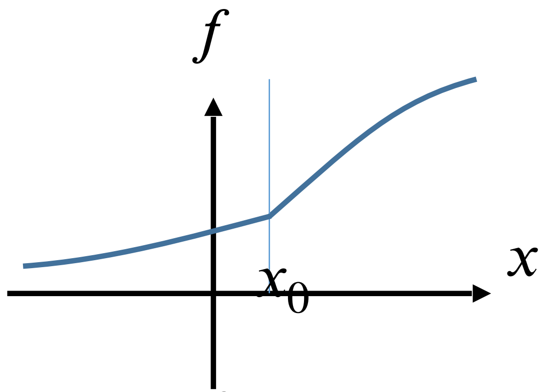
(1)

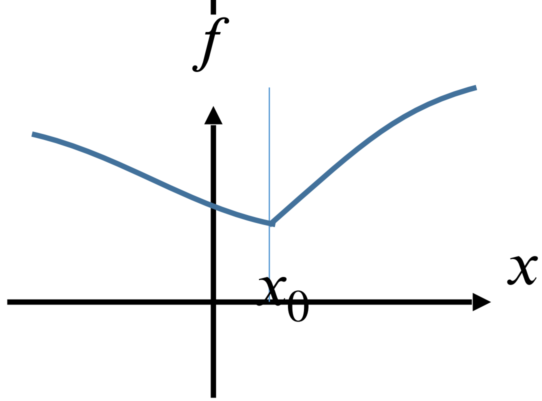


smooth and one extremum

$$e^{iVf(x)} \sim \frac{1}{\sqrt{V}} e^{i\,Vf(x_0)} \delta(x - x_0)$$

(2)





f is continuous

f' is discontinuous at x_0

x_0 need not be extremum

monotonic on each side $x \lessgtr x_0$

$$e^{iVf(x)} \sim \frac{1}{V}\left(\frac{1}{f'(x_0+0)} - \frac{1}{f'(x_0-0)}\right) \cdot e^{i\,Vf(x_0)} \delta(x - x_0)$$

Generalize MPP:

The coupling constants of the low energy effective canonical FT of MM or quantum gravity are automatically adjusted either to minimize the vacuum energy density or to one of the critical points of the history of universe.

Examples

1. QCD θ-parameter

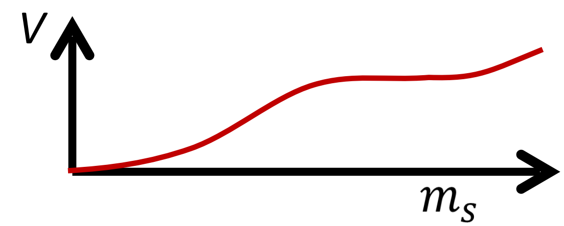
   $\theta = 0$ minimizes the vacuum energy.

2. Cosmological constant

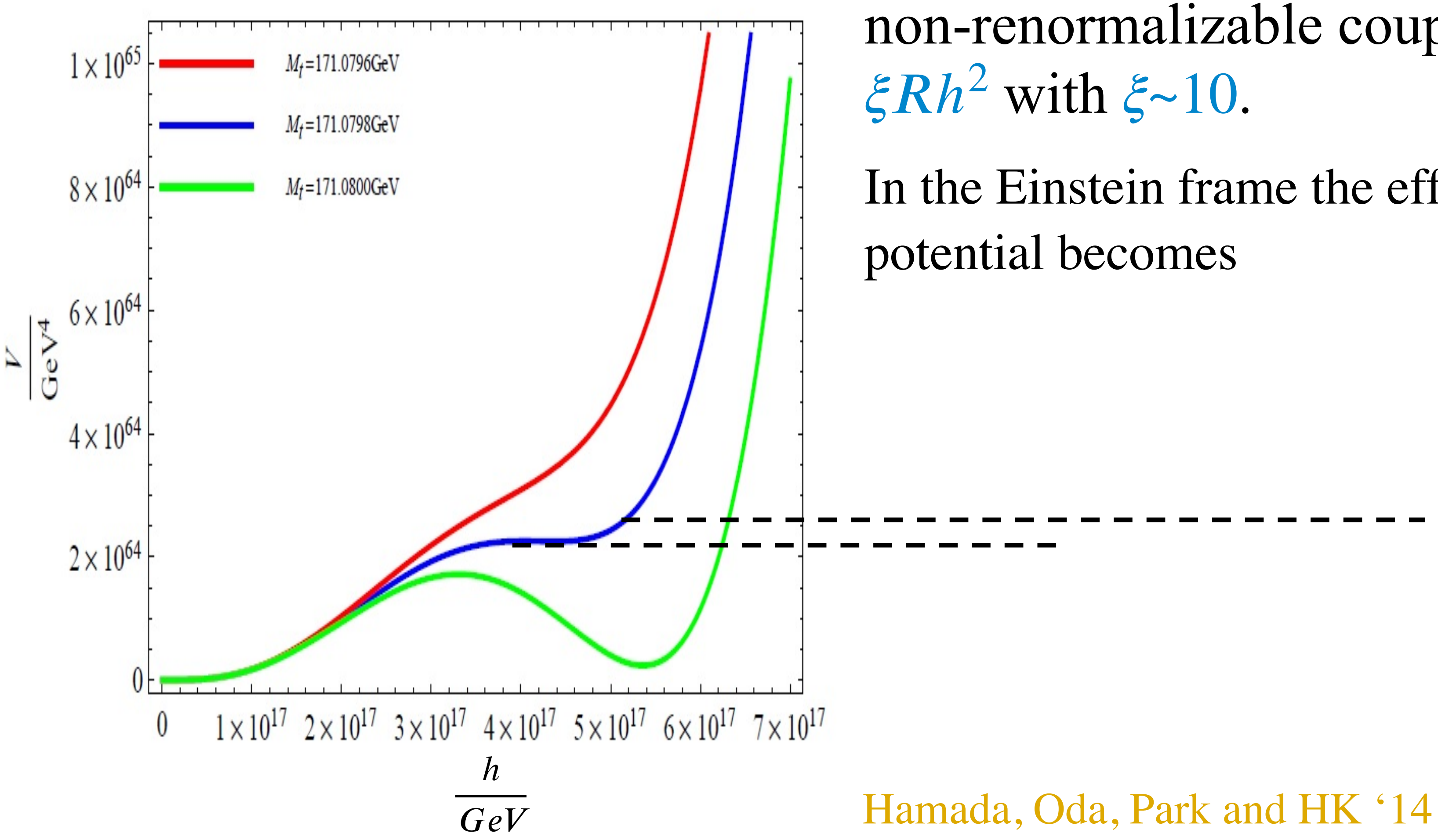
   $\lambda = 0$ is the critical point.

3. Higgs inflation at criticality

   Flat potential is the critical point of the history of universe.



SM Higgs is close to MPP.



non-renormalizable coupling $\xi R h^2$ with $\xi \sim 10$.

In the Einstein frame the effective potential becomes

Hamada, Oda, Park and HK '14
Bezrukov,Shaposhnikov

comment on Green's functions

$$\int \mathscr{D}q f\left(S_i[q] - A_i\right) O[q]$$

$\leftarrow O[q]$: product of local operators

$$= \int \mathscr{D}q \int \prod_i d\alpha_i \tilde{f}(\alpha)\, e^{i\sum_i \alpha_i (S_i[q]-A_i)} O[q]$$

$$= \int \prod_i d\alpha_i w(\alpha) \int \mathscr{D}q e^{i\sum_i \alpha_i S_i[q]} O[q]$$

$$= \int \prod_i d\alpha_i w(\alpha) \frac{\int \mathscr{D}q e^{i\sum_i \alpha_i S_i[q]} O[q]}{\int \mathscr{D}q e^{i\sum_i \alpha_i S_i[q]}} \int \mathscr{D}q e^{i\sum_i \alpha_i S_i[q]}$$

$$= \int \prod_i d\alpha_i w(\alpha) \left\langle O[q] \right\rangle_\alpha \int \mathscr{D}q e^{i\sum_i \alpha_i S_i[q]}$$

$\left\langle O[q] \right\rangle_\alpha$:Ordinary FT with coupling constants α_i .

$$w(\alpha) = \tilde{f}(\alpha)\, e^{-i\sum_i \alpha_i A_i}$$

$\left\langle O[q] \right\rangle_\alpha$ is intensive. $\Rightarrow$ Does not affect $\alpha_i^{(0)}$.