# Avoiding the Ostrogradsky Instability

**Adventures in quantum field theory with higher derivatives**

1) Motivation – higher derivatives and Ostrogradsky

2) A simple model, simply analyzed

3) What would Ostrogradsky say?

4) Time reversal and ghosts – Merlin modes

5) Causality

6) Unitarity

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# My motivation: Quadratic gravity:

My work is with Gabriel Menezes:
arXiv:1712.04468 , arXiv:1804.04980,
arXiv:1812.03603
arXiv:1908.02416, arXiv:1908.04170,
aXiv:2003.09047…
But this understanding builds on the past work of many others

$$S_{\rm quad} = \int d^4x \sqrt{-g} \left[ \frac{2}{\kappa^2} R + \frac{1}{6 f_0^2} R^2 - \frac{1}{2\xi^2} C_{\mu\nu\alpha\beta} C^{\mu\nu\alpha\beta} \right]$$

Renormalizeable QFT for quantum gravity
- the most conservative version of quantum gravity

BUT:

$$R \sim \partial^2 g \qquad\qquad R^2 \sim \partial^2 g \partial^2 g$$

Propagators then have the form

$$iD_F \sim \frac{i}{q^2 - \frac{q^4}{M^2}} = \frac{i}{q^2} - \frac{i}{q^2 - M^2}$$

Higher derivative theories have “issues” and mythology
- Ostrogradsky instability
- negative energy
- unitarity
- causality

## There does have to be an "issue":

Axiomatic field theorists tell us - Källén–Lehmann spectral representation

$$D_F(k) = \int \frac{d^4k}{(2\pi)^4} e^{ik\cdot x} \langle 0|T\phi(x)\phi(0)|0\rangle = \frac{1}{\pi}\int ds \frac{\rho(s)}{k^2 - s + i\epsilon}$$

with $\rho(s)$ positive definite

Implies propagators cannot fall faster than $1/k^2$

$$D_F(k) \sim \frac{1}{k^2}\frac{1}{\pi}\int ds \rho(s)$$

Something sacred must fail

**It is important to understand what fails and how fatal that is.**

# <u>Ostrogradsky</u>

**Deep historical research from Wikipedia**

- Mikhail Vasilyevich Ostrogradsky 1801- 1862
- Russian mathematician
- Educated at Sorbonne, College de France
- work on algebraic functions, calculus of variations

**The Ostrogradsky instability (1850)**

- theories with higher time derivatives
- requires extra canonical coordinates and canonical momenta
- Hamiltonian chosen to reproduce Hamilton's equations
- result is not positive definite – even at low energy
- **to be reviewed in more detail below**

The instability is often used to rule out higher derivative theories

# Effective Field Theory and higher derivatives

Almost all theories have higher derivative corrections

**Example: QED at low energy**
- from vacuum polarization

$$\mathcal{L}_{eff} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{\alpha}{60\pi m_e^2}F_{\mu\nu}\Box F^{\mu\nu} + \ ...$$

This does not cause any instability. **Why not?**

The Ostrogradsky Hamiltonian is not used here
- interaction treated as perturbation
- just use $H_I = -L_I$ (at least when using dim. reg.)

But this does mean that Hamilton's equations do not work

Also cannot use this form at high energy (no K-L problem)

# Explore the physics with a very simple model

**First "normal" version** (i.e. with only two derivatives)

$$\mathcal{L} = \mathcal{L}_\chi + \mathcal{L}_\phi - g\phi\chi^\dagger\chi$$

$$\mathcal{L}_\phi = \frac{1}{2}\left[\partial_\mu\phi\partial^\mu\phi - m^2\phi\phi\right]$$

$$\mathcal{L}_\chi = \partial_\mu\chi^\dagger\partial^\mu\chi - m_\chi^2\chi^\dagger\chi - \lambda(\chi^\dagger\chi)^2$$

Think of $\chi$ as "electron" and $\phi$ as "photon" or "graviton"
I will use $m \ll m_\chi$ and pretend that renormalized $m \to 0$

This is then scalar analog of QED or GR

If $m = 0$ we have the classical wave equation
in long wavelength limit

# Now create the dangerous version with higher derivatives

$$\mathcal{L}_\phi \to \mathcal{L}_{hd}$$

$$\mathcal{L}_{hd} = \frac{1}{2}\left[\partial_\mu\phi\partial^\mu\phi - m^2\phi\phi - \frac{1}{M^2}\Box\phi\Box\phi\right]$$

Here think of M as the Planck mass – beyond our present experiments

**How does QFT treat this?**

Is this stable at low energy? Ostrogradsky says no.

Is there a classical limit?

Is this stable at high energy?

What sacred QFT principle fails?

# First: Low Energy / Classical Limit

**Use path integrals to define the theory**

$$Z_\phi[\chi] = \int [d\phi] e^{i\int d^4x [\mathcal{L}_{hd} - g\phi\chi^\dagger\chi]}.$$

$$\mathcal{L}_{hd} = \frac{1}{2}\left[\partial_\mu\phi\partial^\mu\phi - m^2\phi\phi - \frac{1}{M^2}\Box\phi\Box\phi\right]$$

Now use auxiliary field to remove higher derivative term (*m=0* here)

$$Z_\phi[\chi] = \int [d\phi][d\eta] e^{i\int d^4x [\mathcal{L}(\phi,\eta)]}$$

$$\mathcal{L}(\phi,\eta) = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \eta\Box\phi + \frac{1}{2}M^2\eta^2 - g\phi\chi^\dagger\chi$$

Via Gaussian integration:

$$\frac{1}{2}M^2\eta^2 - \eta\Box\phi = \frac{1}{2}M^2\left(\eta - \frac{1}{M^2}\Box\phi\right)^2 - \frac{1}{2M^2}\Box\phi\Box\phi$$

## Next redefine the field variables

$$\mathcal{L}(\phi,\eta) = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \eta\Box\phi + \frac{1}{2}M^2\eta^2 - g\phi\chi^\dagger\chi$$

Use $\phi(x) = a(x) - \eta(x)$

$$\begin{aligned}\mathcal{L}(a,\eta) &= \left[\frac{1}{2}\partial_\mu a\partial^\mu a - ga\chi^\dagger\chi\right] \\ &- \left[\frac{1}{2}\partial_\mu\eta\partial^\mu\eta - \frac{1}{2}M^2\eta^2 - g\eta\chi^\dagger\chi\right]\end{aligned}$$

This totally decouples the fields

$$\begin{aligned}Z_\phi[\chi] &= \int[da]e^{i\int d^4x\left[\frac{1}{2}\partial_\mu a\partial^\mu a - ga\chi^\dagger\chi\right]} \\ &\times \int[d\eta]e^{-i\int d^4x\left[\frac{1}{2}\partial_\mu\eta\partial^\mu\eta - \frac{1}{2}M^2\eta^2 - g\eta\chi^\dagger\chi\right]} \\ &= Z_a \times Z_\eta\end{aligned}$$

Here $Z_a$ is just normal PI, and $Z_\eta$ is complex conjugate of normal PI

**We can do the $\eta$ path integral, as a usual Gaussian integral**

Add $-\epsilon \int d^4x\, \phi^2$ for convergence, then

$$\eta'(x) = \eta(x) - \int d^4x iD_{-F}(x-y)\ \chi^\dagger(y)\chi(y)$$

With $iD_{-F}(x-y) = \int \frac{d^4k}{(2\pi)^4} \frac{-i}{k^2 - M^2 - i\epsilon}$

This result in:

$$Z_\eta = N e^{\int d^4x d^4y \frac{1}{2} g\chi^\dagger(x)\chi(x)\ iD_{-F}(x-y)\ g\chi^\dagger(y)\chi(y)}$$

At low energy, this becomes a contact interaction

$$Z_\eta = N e^{i\int d^4x \frac{g^2}{2M^2}[\chi^\dagger(x)\chi(x)]^2}$$

The result is just a shift in $\lambda$ in the $\chi$ interaction

$$\lambda \to \lambda' = \lambda - \frac{g^2}{2M^2}$$

# <u>Low energy limit is a normal theory</u>

- The original normal theory with a shifted value of $\lambda$

No sign of Ostrogradsky instability

This has a normal classical limit

Note: really not $\hbar \to 0$ because $\hbar$ is a constant.
- Classical limit is kinematics where $\hbar$ is not important
- here low energy compared to "electron" Compton wavelength

**This is already sufficient evidence to refute the Ostrogradsky conclusion**

## **Interpretation to be developed – time reversed "ghost"**

Time reversal is anti-unitary – involves complex conjugation
Lagrangian is invariant
Path integral changes sign

This will lead to interpretation of the "ghost" as time reversed field
- *Merlin mode*

# What about high energy?

- there is certainly something "funny" there
- high mass pole in the propagator with the wrong signature

$$iD_F \sim \frac{i}{q^2 - \frac{q^4}{M^2}} = \frac{i}{q^2} - \frac{i}{q^2 - M^2}$$

Here is an extremely important effect – **decay to light states**

$$iD(q^2) = \frac{i}{q^2 - m^2 + i\epsilon - \frac{q^4}{M^2} + \Sigma(q)} \quad .$$

The imaginary part is fixed by a normal vacuum polarization calculation

$$\Sigma_f(q) = -\frac{g^2}{32\pi^2}\int_0^1 dx \log\left[\frac{m_\chi^2 - x(1-x)(q^2+i\epsilon)}{m_\chi^2}\right]$$

At high $q^2$ this has

$$\mathrm{Im}\Sigma \sim \frac{g^2}{32\pi} \equiv \gamma$$

which is positive

Near high mass pole

$$
\begin{aligned}
iD(q) \;&\sim\; \frac{i}{q^2 - \frac{q^4}{M^2} + \mathrm{Re}\Sigma + i\gamma} \\
&\sim\; \frac{-i}{q^2 - \bar{M}^2 - i\gamma} \quad .
\end{aligned}
$$

There are **two** minus sign differences here
- it is the complex conjugate of a usual resonance propogator

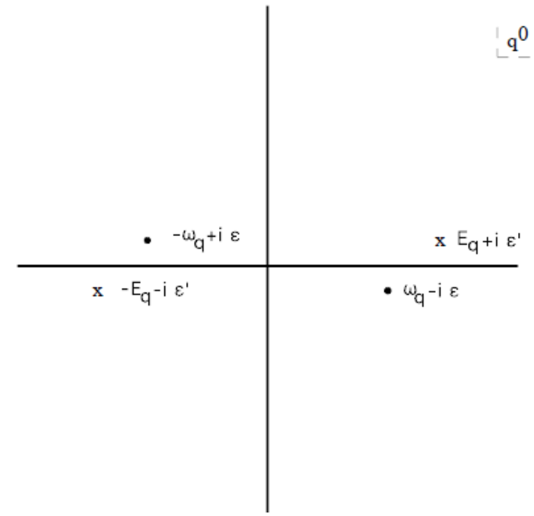
# Propagator – time orderings

$$iD_F(x) = D_F^{\text{for}}(x)\theta(t) + D_F^{\text{back}}(x)\theta(-t)$$

Note **energy flow**, and also decay lifetime

$$D^{\text{for}}(t,\vec{x}) = \int \frac{d^3q}{(2\pi)^3}\left[\frac{e^{-i(\omega_q t - \vec{q}\cdot\vec{x})}}{2\omega_q} - \frac{e^{i(E_q t - \vec{q}\cdot\vec{x})}}{2(E_q + i\frac{\gamma}{2E_q})}e^{-\frac{\gamma t}{2E_q}}\right]$$

$$D^{\text{back}}(t,\vec{x}) = \int \frac{d^3q}{(2\pi)^3}\left[\frac{e^{i(\omega_q t - \vec{q}\cdot\vec{x})}}{2\omega_q} - \frac{e^{-i(E_q t - \vec{q}\cdot\vec{x})}}{2(E_q + i\frac{\gamma}{2E_q})}e^{-\frac{\gamma |t|}{2E_q}}\right]$$

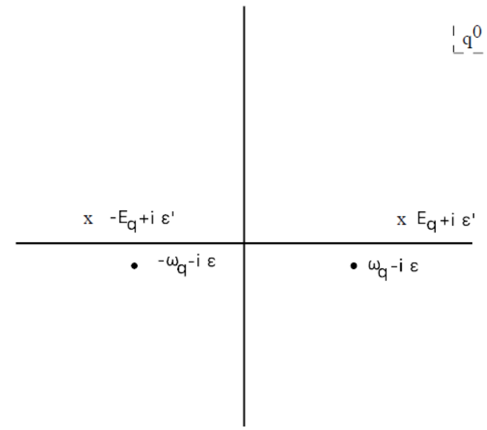



Massive state is unstable – decay rate $\sim \gamma$
- decay, not exponential growth

# “Retarded Green function”:

Consider propagator with retarded BC:

$$\log\left(-\left[(q_0+i\epsilon)^2-\vec{q}^2\right]\right)=\log\left(-q^2-i\epsilon q_0\right)=\log|q^2|-i\pi\theta(q^2)\left(\theta(q_0)-\theta(-q_0)\right)$$




Again propagation in both directions:

$$D_{\rm ret}(t>0,\vec{x})=D^{(0)}_{\rm ret}(t>0,\vec{x})$$

Backwards perturbations have finite lifetime:

$$D_{\rm ret}(t<0,\vec{x})\equiv D^{<}_{\rm ret}(t,\vec{x})=i\int\frac{d^3q}{(2\pi)^3}\left[\frac{e^{-i(E_qt-\vec{q}\cdot\vec{x})}}{2(E_q+i\frac{\gamma}{2E_q})}e^{-\frac{\gamma|t|}{2E_q}}-\frac{e^{i(E_qt-\vec{q}\cdot\vec{x})}}{2(E_q-i\frac{\gamma}{2E_q})}e^{-\frac{\gamma|t|}{2E_q}}\right]$$

No growing modes

$$h_{\mu\nu}(t,x)=\int d^3x'\left[\int_{-\infty}^{t}dt'D^{(0)}_{\rm ret}(t-t',x-x')+\int_t^{\infty}dt'D^{<}_{\rm ret}(t-t',x-x')\right]J_{\mu\nu}(t',x')$$

# **Positive energy**

To produce or detect massive state use $\bar{\chi}\chi \to M \to \bar{\chi}\chi$

$$\mathcal{M} = g^2 D(q) \sim \frac{-g^2}{q^2 - \bar{M}^2 - i\gamma}$$

Squared matrix element is same as usual BW

Incoming and outgoing particles carry positive energy
- implies that this is a positive energy resonance

# What would Ostrogradsky say?

Extra canonical coordinates and momenta

$$\begin{aligned}\phi_1 &= \phi \\ \phi_2 &= \dot{\phi}\end{aligned} \qquad\qquad \begin{aligned}\pi_1 &= \frac{\partial \mathcal{L}}{\partial \dot{\phi}} - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \ddot{\phi}} = \left(\frac{\Box + M^2}{M^2}\right)\dot{\phi} \\ \pi_2 &= \frac{\partial \mathcal{L}}{\partial \ddot{\phi}} = -\frac{\Box}{M^2}\phi \ .\end{aligned}$$

Hamiltonian

$$\mathcal{H}(\phi_1, \phi_2, \pi_1, \pi_2) = \pi_1 \dot{\phi}_1 + \pi_2 \dot{\phi}_2 - \mathcal{L}$$

But have to eliminate $\ddot{\phi}$ in favor of the coordinates and momenta

$$\ddot{\phi} = \nabla^2 \phi - M^2 \pi_2$$

This leads to the final Hamiltonian

$$\mathcal{H} = \pi_1 \phi_2 + \pi_2 \left(\nabla^2 \phi - M^2 \pi_2\right) - \mathcal{L}(\phi_1, \phi_2, \nabla^2 \phi - M^2 \pi_2)$$

The first term is the Ostrogradsky instability

- $\pi_1$ and $\phi_2$ can have either sign
- this is the only place where $\pi_1$ enters the Hamiltonian

## **Why these choices?**

Chosen to reproduce Hamilton's equations

$$\dot{\phi}_1 = \frac{\partial \mathcal{H}}{\partial \pi_1}$$
$$\dot{\phi}_2 = \frac{\partial \mathcal{H}}{\partial \pi_2}$$

The Euler Lagrange equation follows from

$$\dot{\pi}_1 = -\frac{\partial \mathcal{H}}{\partial \phi_1}$$

But from QFT point of view, this is not a natural construction

# Need for caution:

**All of our equivalences are based on normal theories**

Should Hamilton's equations be modified in HD theories?

Are path integral and canonical quantization equivalent with HD theories?

Is the free field theory representative of the full theory?

**Also we know that classical instabilities do not have to be quantum instabilities**

- Dirac field has classical instability
- solved by change of commutation rules

**Usual way we teach/discuss theories:**

1) Classical physics and solutions
2) Canonical Hamiltonian quantization of free field theory
3) Add interactions
4) Repeat with Lagrangian Path Integrals

**Here – reverse pathway:**

1) Start with Lorentzian Lagrangian Path Integral
2) Include interactions with matter
3) Then, analyze theory
4) Limits to standard EFT at low energy (and normal classical physics)

Reverse pathway is like the approach to Electroweak theory

- need to identify spectrum correctly first

# What about canonical quantization?

We have avoided it by using PI

But canonical quantization does exist for these theories
- indefinite metric quantization of Lee, Wick
- also Salvio, Strumia  and Raidal, Veerme
- PT theory of Bender and Mannheim

Modified quantization rules lead to positive energy spectrum

# Our heuristic version:

- see originals for more careful treatment

$$\mathcal{L}(a,\eta) = \left[\frac{1}{2}\partial_\mu a \partial^\mu a - g a \chi^\dagger \chi\right] - \left[\frac{1}{2}\partial_\mu \eta \partial^\mu \eta - \frac{1}{2}M^2\eta^2 - g\eta\chi^\dagger\chi\right]$$

One can see the dangerous sign in forming the Hamiltonian

But canonical commutators also change

$$\pi_\eta = \frac{\partial \mathcal{L}}{\partial \dot{\eta}} = -\dot{\eta}$$

$$[\eta(x,t), \pi_\eta(x',t)] = i\hbar\delta^3(x-x')$$

which implies the complex conjugate of the usual relation

$$[\eta(x,t), \dot{\eta}(x',t)] = -i\hbar\delta^3(x-x') \quad .$$

To solve this with the usual field decomposition we need

$$[a(p), a^\dagger(p')] = -\delta^3(p-p')$$

This tells us that the Hamiltonian actually has positive energy states

$$H_\eta = -\int \frac{d^3p}{(2\pi)^3}\sqrt{p^2+M^2}\; a^\dagger(p)a(p) \qquad\qquad |p'\rangle = a^\dagger|0\rangle.$$

# Basic conclusion on Ostrogradsky

**Ostrogradsky construction is not that of QFT**

Path integral can define the theory without Hamiltonian quantization
- finds decaying massive states
- but low energy/classical limit appears normal in at least some cases

Canonical quantization needs modified commutators
- but methods exist for positive energy states

## But recall: there does have to be an "issue":

Axiomatic field theorists tell us - Källén–Lehmann spectral representation

$$D_F(k) = \int \frac{d^4k}{(2\pi)^4} e^{ik\cdot x} \langle 0|T\phi(x)\phi(0)|0\rangle = \frac{1}{\pi}\int ds \frac{\rho(s)}{k^2 - s + i\epsilon}$$

with $\rho(s)$ positive definite

Implies propagators cannot fall faster than $1/k^2$

$$D_F(k) \sim \frac{1}{k^2}\frac{1}{\pi}\int ds\rho(s)$$

**Something sacred must fail**

# Causality

Known since Lee-Wick and Coleman that such propagators lead to micro-causality violation

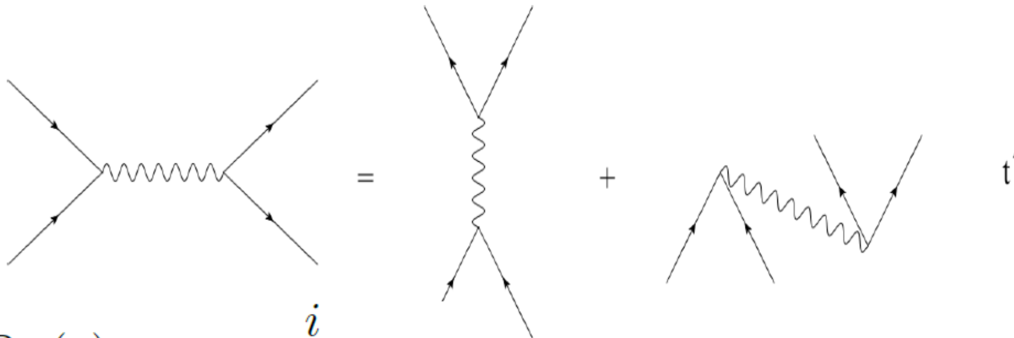
Traced to backwards-in-time propagation
- dueling **arrows of causality**

But appears limited to time scales proportional to lifetime

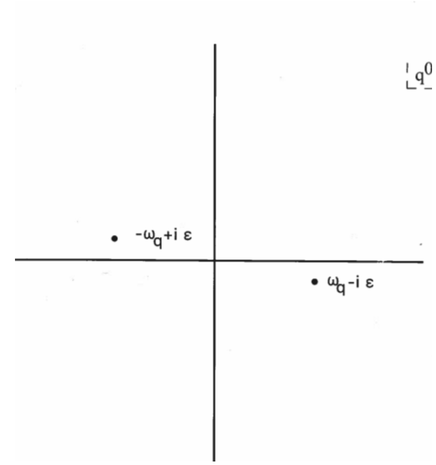
For gravity this is inverse Planck scale

# More on causality

Causality is not really “cause before effect”




$$iD_F(q) = \frac{i}{q^2 - m^2 + i\epsilon}$$




Decompose into time orderings:

$$iD_F(x) = D_F^{\rm for}(x)\theta(t) + D_F^{\rm back}(x)\theta(-t)$$

Positive energies propagate forward in time
- backwards propagation is “negative energy”

But backward-in-time propagation shielded by uncertainty principle
$\Delta t \sim 1/\Delta E$

# Operators commute for spacelike separation

$$[\mathcal{O}(x), \mathcal{O}(x')] = 0 \quad \text{for} \quad (x - x')^2 < 0$$

Note: metric is (+,-,-,-)

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## Use of Causality Conditions in Quantum Theory

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(Received May 24, 1954)

The limitations on scattering amplitudes imposed by causality requirements are deduced from the demand that the commutator of field operators vanish if the operators are taken at points with space-like separations. The problems of the scattering of spin-zero particles by a force center and the scattering of photons by a quantized matter field are discussed. The causality requirements lead in a natural way to the well-known dispersion relation of Kramers and Kronig. A new sum rule for the nuclear photoeffect is derived and the scattering of photons by nucleons is discussed.

This requires negative energy part of propagator to accomplish

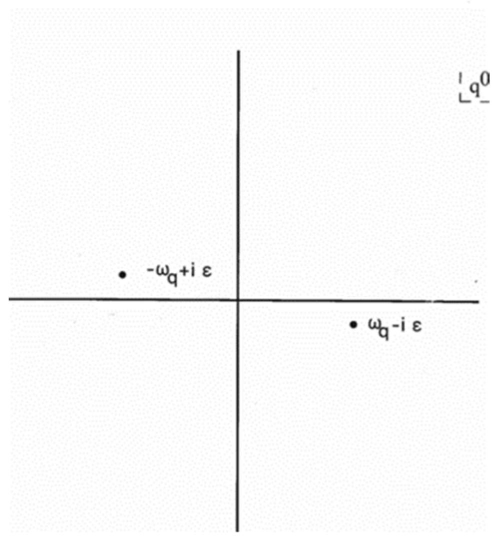
# But also – Arrow of Causality

What determines past lightcone and future lightcone?
- and why do all particles share this?

This comes from the $i\epsilon$

$$iD_F(q) = \frac{i}{q^2 - m^2 + i\epsilon}$$

Determines that positive energy propagates forward in time



# What if we used $e^{-iS}$ instead of $e^{iS}$?

Consider generating functions:

$$Z_\pm[J] = \int [d\phi] e^{\pm i S(\phi, J)}$$

$$= \int [d\phi] e^{\pm i \int d^4x [\frac{1}{2}(\partial_\mu \phi \partial^\mu \phi - m^2 \phi^2) + J\phi]} .$$

Need to make this better defined – add

$$e^{-\epsilon \int d^4 x \phi^2 / 2}$$

Solved by completing the square:

$$Z_\pm[J] = Z[0] \exp\left\{ -\frac{1}{2} \int d^4x d^4y J(x)\ iD_{\pm F}(x-y) J(y) \right\}$$

Yield propagator with specific analyticity structure

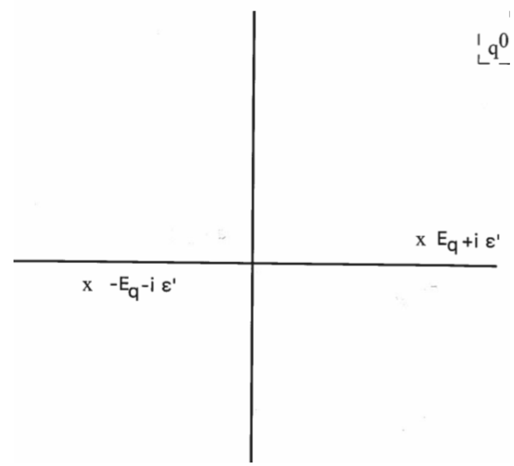
$$iD_{\pm F}(x-y) = \int \frac{d^4q}{(2\pi)^4} e^{-iq\cdot(x-y)} \frac{\pm i}{q^2 - m^2 \pm i\epsilon}$$

## Result is time-reversed propagator

$$iD_{-F}(x) = D^{\text{for}}_{-F}(x)\theta(t) + D^{\text{back}}_{-F}(x)\theta(-t)$$

"Positive energy" propagates backwards in time

$$D^{\text{back}}_{-F}(x) = \int \frac{d^3q}{(2\pi)^3 2E_q} e^{-i(E_q t - \vec{q}\cdot\vec{x})}$$




Use of this generating functional yields time reversed scattering processes

**Opposite arrow of causality**

# Time reversal is anti-unitary

Lagrangian can be invariant, but PI is not

$$Z_+[J] \to Z_-[J]$$

Note: Also can be found in canonical quantization
Changes

$$[\phi(t,x), \pi(t,x')] = i\hbar\delta^3(x-x')$$

to

$$[\phi(t,x), \bar{\pi}(t,x')] = -i\hbar\delta^3(x-x') \quad \text{with} \quad \bar{\pi} = \frac{\partial \mathcal{L}}{\partial(\partial_\tau \phi)}.$$

## “Arrow of time”:

The conventional wisdom is actually wrong here!

Typical statement:
*"The laws of physics at the fundamental level don’t distinguish between the past and the future."*

**But this is not correct**

The laws of quantum physics does have an arrow of causality

Buried in the factors of i in the quantization procedures

Our time convention uses $Z_+$
- if reverse time convention used, use $Z_-$

**Note**: Arrow of thermodynamics follows arrow of causality

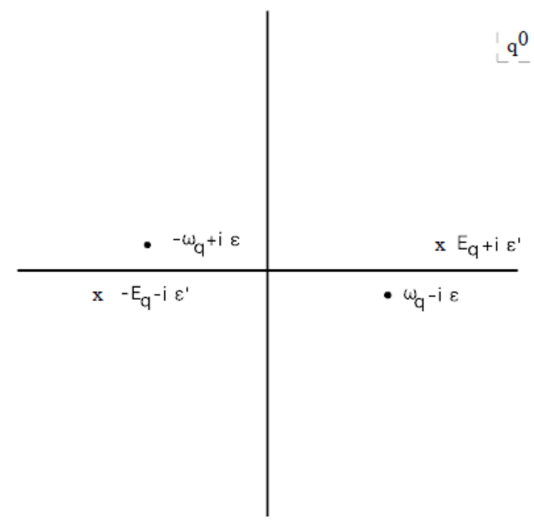
# Recall our propagator – time orderings

$$iD_F(x) = D_F^{\rm for}(x)\theta(t) + D_F^{\rm back}(x)\theta(-t)$$

Note **energy flow**, and also decay lifetime

$$D^{\rm for}(t,\vec{x}) = \int \frac{d^3q}{(2\pi)^3}\left[\frac{e^{-i(\omega_q t - \vec{q}\cdot\vec{x})}}{2\omega_q} - \frac{e^{i(E_q t - \vec{q}\cdot\vec{x})}}{2(E_q + i\frac{\gamma}{2E_q})}e^{-\frac{\gamma t}{2E_q}}\right]$$

$$D^{\rm back}(t,\vec{x}) = \int \frac{d^3q}{(2\pi)^3}\left[\frac{e^{i(\omega_q t - \vec{q}\cdot\vec{x})}}{2\omega_q} - \frac{e^{-i(E_q t - \vec{q}\cdot\vec{x})}}{2(E_q + i\frac{\gamma}{2E_q})}e^{-\frac{\gamma |t|}{2E_q}}\right]$$




Massive state is unstable – decay rate $\sim \gamma$
- decay, not exponential growth

# Resonance production

Recall positive energy argument using $\bar{\chi}\chi \to M \to \bar{\chi}\chi$

$$\mathcal{M} = g^2 D(q) \sim \frac{-g^2}{q^2 - \bar{M}^2 - i\gamma}$$

Incoming and outgoing particles carry positive energy
- implies that this is a positive energy resonance

But now we know that positive energy propagates resonance backward!

**Interpretation:**

This is different from normal resonance

$$iD_F \sim_{q^2\sim m^2} = \frac{i}{q^2 - m^2 + i\gamma_m} \sim \frac{i}{q^2 - \left(m_r - i\frac{1}{2}\Gamma\right)^2}$$

Here we have

$$\sim \frac{-i}{q^2 - M^2 - i\gamma_M}$$

This is **time-reversed** version of a resonance propagator

- time reversal is anti-unitary

Still corresponds to decaying particle

Important for unitarity – **imaginary parts are the same**

$$iD(q) \sim \frac{Zi}{q^2 - m^2 + iZ\gamma} \qquad \mathrm{Im}[D(q)] \sim \frac{-\gamma}{(q^2 - m^2)^2 + \gamma^2}$$

# Merlin modes:

-Merlin (the wizard in the tales of King Arthur) ages backwards

*"Now ordinary people are born forwards in Time, if you understand what I mean, and nearly everything in the world goes forward too. (...) But I unfortunately was born at the wrong end of time, and I have to live backwards from in front, while surrounded by a lot of people living forwards from behind."*

T. H. White *Once and Future King*

Note, there is a key distinction with the usual nomenclature "ghosts"

- ghost is anything with a minus sign in the numerator
- Fadeev-Popov ghosts have the usual $+i\epsilon$ in the denominator
- Merlin modes have a crucial $-i\gamma$ in the denominator

# Dueling arrows of causality

Quartic propagators have opposing arrows

$$iD(q^2) \sim \frac{i}{q^2 + i\epsilon} \qquad \text{vs} \quad \sim \frac{-i}{q^2 - M^2 - i\gamma_M}$$

**Who wins?**

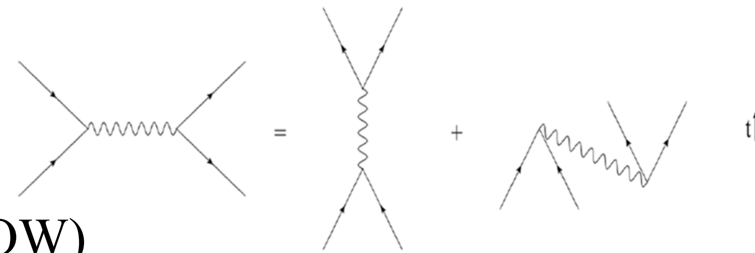
-massive state decays

-stable states win

- macro causality is determined by the stable states

## Phenomenology

Lee, Wick
Coleman
Grinstein, O'Connell, Wise
Alvarez, Da Roid, Schat, Szynkman



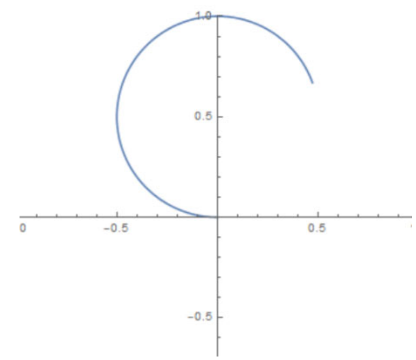
**Vertex displacements**: (ADSS)
- look for final state emergence
- before beam collision

**Form wavepackets – early arrival** (LW, GOW)
- wavepacket description of scattering process
- some components arrive at detector early

**Resonance Wigner time delay reversal**
- normal resonances counterclockwise on Argand diagram

- Merlin modes are clockwise resonance

$$\Delta t \sim \frac{\partial \delta}{\partial E} \sim> 0$$



**For gravity, all are Planck scale**
- no conflict with experiment

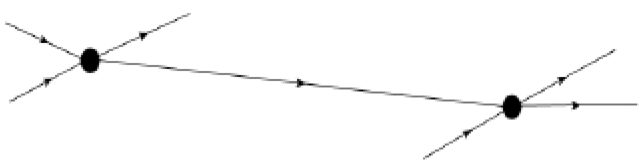
# Living with Causality Uncertainty

Wavepackets are an idealization:
 -really formed by previous interactions

Likewise beam construction from previous scattering
 - and measurement due to final scattering

Type equation here.

The timing of scattering will become **uncertain**



**But causal uncertainty is likely a general property of quantum gravity**

JFD with G. Menezes
2106.05912

# <u>Unitarity of unstable particles:</u>

$$\langle f|T|i\rangle - \langle f|T^\dagger|i\rangle = i\sum_j \langle f|T^\dagger|j\rangle\langle j|T|i\rangle$$

**Who counts in unitarity relation?**

- Veltman 1963
- **only stable particles count**
- they form asymptotic Hilbert space
- **do not** make any cuts on unstable resonances

UNITARITY AND CAUSALITY IN A RENORMALIZABLE FIELD THEORY WITH UNSTABLE PARTICLES

M. VELTMAN *)

This looks funny from free-field quantization

- interaction removes states from the Hilbert space

Also, we know some states are almost stable

- can treat them as essentially stable
- Narrow Width Approximation (NWA)

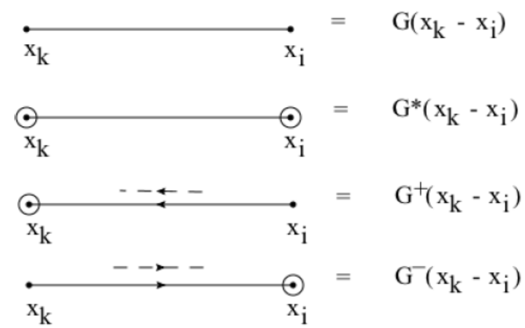
But of course, Veltman is correct

# Formal proof of unitarity with unstable ghosts

**With G. Menezes arXiv:1908.02416 in PRD**

**Follows Veltman**:

- circling rules
- largest time equation
- turns into derivation of cutting rules




Only difference is **energy flow**

$$
\begin{aligned}
-iG^*(x-x') &= \Theta(x_0-x_0')G^-(x-x') + \Theta(-x_0+x_0')G^+(x-x') \\
-i\widetilde{G}^*(x-x') &= \Theta(x_0-x_0')\widetilde{G}^-(x-x') + \Theta(-x_0+x_0')\widetilde{G}^+(x-x') \\
-i\widetilde{G}^*_{\rm GH}(x-x') &= \Theta(x_0-x_0')\widetilde{G}^+_{\rm GH}(x-x') + \Theta(-x_0+x_0')\widetilde{G}^-_{\rm GH}(x-x')
\end{aligned}
$$

Important point - all steps in Minkoswki space
- no analytic continuation employed

**Formalizes early work by Lee-Wick**

# Explicit example – resonance - $\bar{\chi}\chi \to M \to \bar{\chi}\chi$

S-wave elastic scattering partial wave

$$T_0(s) = \frac{g^2}{32\pi} D(s) \qquad iD(q^2) = \frac{i}{q^2 - m^2 + i\epsilon - \frac{q^4}{M^2} + \Sigma(q)}$$

Recall width:

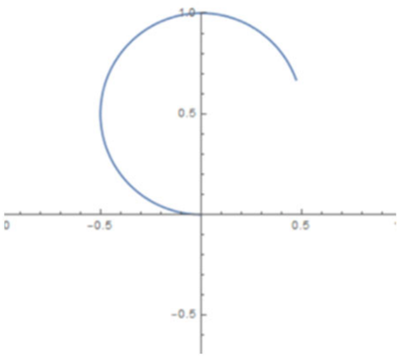
$$\gamma = \frac{g^2}{32\pi}$$

Near resonance this takes the form

$$T_0 = \frac{-g^2}{32\pi} \frac{1}{q^2 - \bar{M}^2 - i\frac{g^2}{32\pi}}$$

This satisfies elastic unitarity

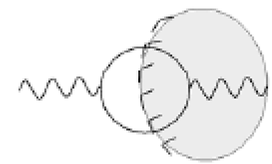
$$\mathrm{Im} T_0 = |T_0|^2$$



# Why this works: Cutkosky cutting rules

Obtain discontinuity by replacing propagator with:

$$\frac{i}{q^2 - m^2 + i\epsilon} \to 2\pi\delta(q^2 - m^2)\theta(q_0)$$



Also on far side of cut, use:

$$\frac{i}{q^2 - m^2 + i\epsilon} \to \frac{-i}{q^2 - m^2 - i\epsilon}$$

Example – self energy

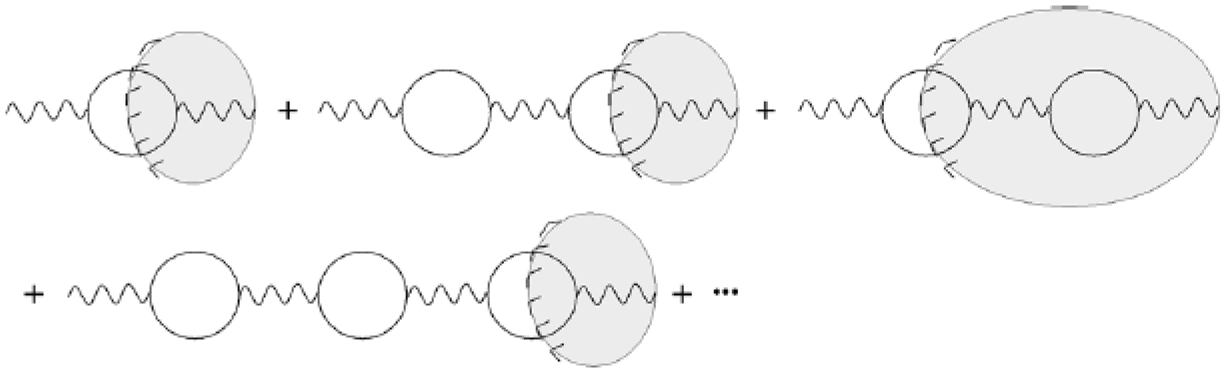
$$\mathrm{Disc}_2\ \Sigma(q) = \frac{\kappa^2 q^4 (N+1)}{2} \int \frac{d^4k}{(2\pi)^4}\ 2\pi\delta(k^2)\theta(k_0)\ 2\pi\delta((q-k)^2)\theta((q-k)_0)$$

Can repackage this:

$$\mathrm{Disc}_2\ \Sigma(q) = 2q\Gamma_2(q)$$

The discontinuity is equivalent to the decay width at $q^2$

## **Cuts in a resonance propagator**:

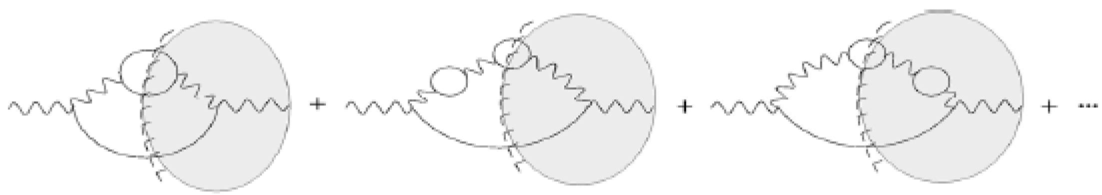



Bubble sum on each side of propagator:
- will c.c. propagators on the far side

$$\text{Disc}\ D(q) = D(q)\ 2q\Gamma_2(q)\ D^*(q) = -2\ \text{Im}[D(q)]$$

This is true no matter if normal resonance or Merlin modes
- imaginary parts are the same

# Three particle cut = resonance + stable cut



$$\mathrm{Disc}_3\ \Sigma(q) = \kappa^2 q^4 \int \frac{d^4k_1}{(2\pi)^4}\frac{d^4k_2}{(2\pi)^4}\frac{1}{(q-k_1)^2 - \frac{\kappa^2(q-k_1)^4}{2\xi^2}}\frac{(N+1)\kappa^2(q-k_1)^2}{2}$$
$$\times\ 2\pi\delta(k_1^2)\theta(k_{10})\ 2\pi\delta(k_2^2)\theta(k_{20})\ 2\pi\delta((q-k_1-k_2)^2)\theta((q-k_1-k_2)_0)\frac{1}{(q-k_1)^2 - \frac{\kappa^2(q-k_1)^4}{2\xi^2}}$$

Identify matrix element

$$\mathcal{M}_3 = \kappa q^2 \kappa (q-k_1)^2 D(q-k_1)$$

and play similar games, to get expected unitarity relation

$$\mathrm{Disc}_3\ \Sigma(q) = 2q\Gamma_3(q)$$

Again result is independent of type of resonance

**Bottom line**: discontinuities come from cuts on stable particles

## Narrow width approximation

Discontinuity in propagator was due to on-shell states only

$$\text{Disc } D(q) = D(q)\ 2q\Gamma(q)\ D^*(q) = \frac{2q\Gamma(q)}{(q^2 - m_r^2)^2 + (m_r\Gamma(q))^2}$$

But when Γ is small, this is highly peaked on the resonance,
Use:

$$\delta(x) = \lim_{\epsilon \to 0} \frac{1}{\pi} \frac{\epsilon}{x^2 + \epsilon^2}$$

Limits to usual cutting rule:

$$\lim_{\Gamma \to 0} \text{Disc } D(q) = 2\pi\delta(q^2 - m_r^2)$$

In “three particle cut” , this is equiv. decay to resonance plus stable

Again, this result is independent of normal or Merlin resonance

# Lessons ala Veltman

**Physics**:
Cuts for resonances actually are through the stable particles

Resonances do not go on-shell

**Math**:
The $i\gamma$ quickly overwhelms the $i\epsilon$

In the end, this is what Veltman 1963 shows

**Think**: LSZ

$$\langle b|S|a\rangle = \left[i\int d^4x_1 e^{-ip_1\cdot x_1}(\Box_1 + m^2)\right]\cdots\left[i\int d^4x_n e^{ip_n\cdot x_n}(\Box_n + m^2)\right]\langle\Omega|T\{\phi(x_1)\cdots\phi(x_n)\}|\Omega\rangle$$

# Heuristic proof of unitarity

Unitarity works with stable particle as external states

Cuts through stable particle loops same for normal and Merlin resonances

Both normal states and Merlin resonances can be in same propagator

Veltman proved normal resonances satisfy unitarity to all orders

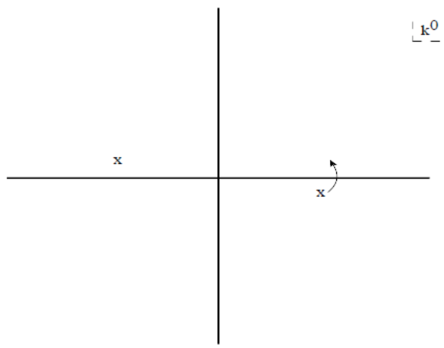
The Merlins will then also satisfy unitarity

# Narrow Width Approximation with Merlin modes

This path follows M. Schwartz: QFT +SM

$$i\mathcal{M} = -\int \frac{d^4k}{(2\pi)^4} \frac{i}{(k-q)^2 + i\epsilon} \frac{-i}{k^2 - m^2 - i\gamma}$$

Normal




Convert $D_F$ to advanced propagator

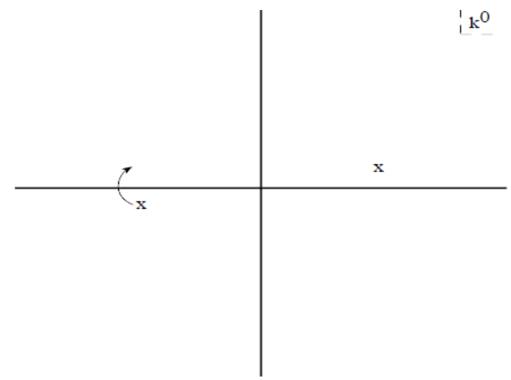
$$\frac{i}{k^2 + i\epsilon} = iD_A(k) + \frac{\pi}{\omega_k}\delta(k_0 - \omega_k)$$

Product of advanced propagators vanishes

Play some games and pick out Im part

$$\mathrm{Im}[\mathcal{M}] = -\int \frac{d^4k}{(2\pi)^4} \left[ \pi\delta((k-q)^2)\ \frac{\pi}{E_k}\delta(k_0 - E_k) - \frac{\pi}{\omega_k}\delta(k_0 - q_0 - \omega_k)\ \pi\delta(k^2 - m^2) \right]$$

Ghost



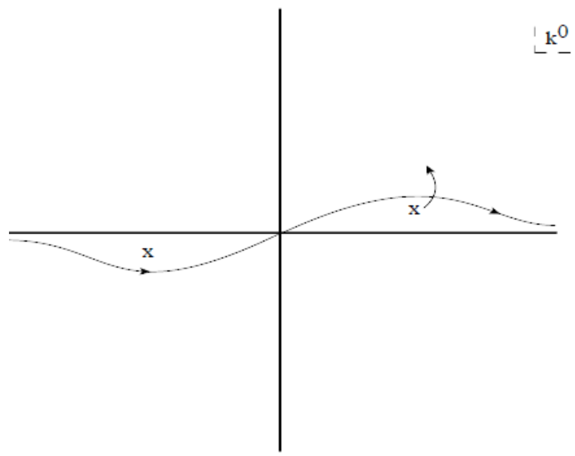

Take same path

$$\frac{-i}{k^2 - m^2 - i\gamma} = -iD_{\text{“}A\text{”}}(k) + \frac{\pi}{E_k}\delta(k_0 + E_{k-p})$$

But delta functions cannot be satisfied

$$\mathrm{Im}[\mathcal{M}] = -\int \frac{d^4k}{(2\pi)^4} \left[ \pi\delta((k-q)^2)\ \frac{\pi}{E_k}\delta(k_0 + E_k) - \frac{\pi}{\omega_k}\delta(k_0 - q_0 - \omega_k)\ \pi\delta(k^2 - m^2) \right]$$

## Lee-Wick contour




Contour goes around poles
Returns to normal discontinuity as $\gamma$ goes to zero

Compatible with usual Wick rotation

Treating ghost like a normal particle requires LW contour

# More needed:

Full field theory is not well-understood
- how to do higher order calculations?

Cutkosky et al
CLOP

More detailed explicit calculations
- Gabriel has one not yet published
- higher order loops

Connection to unitarity-based calculations
- unitarity techniques with unstable particles

Lattice simulations?
- but Euclidean vs. Lorentzian

Etc…

## **Summary**

Higher derivative theories potentially viable

Quantum physics can bypass Ostrogradsky instability

- Ostrogradsky Hamiltonian need not be the quantum Hamiltonian

But still unusual features

- $e^{-iS}$ $vs$ $e^{iS}$ in PI
- indefinite metric quantization rules
- Merlin modes

Violation of microcausality

Unitarity satisfied by using only the stable modes

Full QFT not developed completely

## Quick comments on quadratic gravity:

Quadratic gravity is a renormalizeable quantum field theory

The spin-two propagator has a high-mass Merlin resonance
- similar behavior to simple example above

**Positive features**:
- massless graviton identified through pole in propagator
- ghost resonance decays – does not appear in spectrum
- seems stable under perturbations
- unitarity with only stable asymptotic states
- LW contour as shortcut via narrow width approximation

**Most unusual feature**:
- causality violation/uncertainty near Planck scale

More work needed, but appears as a viable option for quantum gravity