Dark matter stability at fixed points of a modular A(4) symmetry

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09th Sec. 2021 at KIAS

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1.Why modular symmetry?

To predict quark and lepton mixings and masses with more sophisticated way due to a specific Yukawa structure!

Example of traditional A4 model:

Neutrino sector(Weinberg ope.)

$$m_{\nu LL} \sim y_1 \begin{pmatrix} 2\langle\phi_{\nu 1}\rangle & -\langle\phi_{\nu 3}\rangle & -\langle\phi_{\nu 2}\rangle \\ -\langle\phi_{\nu 3}\rangle & 2\langle\phi_{\nu 2}\rangle & -\langle\phi_{\nu 1}\rangle \\ -\langle\phi_{\nu 2}\rangle & -\langle\phi_{\nu 1}\rangle & 2\langle\phi_{\nu 3}\rangle \end{pmatrix} + y_2\langle\xi\rangle \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} + y_3 \underbrace{\langle \xi' \rangle}_{1'} \boxed{\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}}$$

$\mathbf{3_L} \times \mathbf{3_L} \times \mathbf{3_{flavon}} \rightarrow \mathbf{1}$ $\quad\quad$ $\langle\phi_{\nu 1}\rangle = \langle\phi_{\nu 2}\rangle = \langle\phi_{\nu 3}\rangle$ Assumptions

$$M_\nu = \boxed{a \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + b \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} + c \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}} + d \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

Tri-bi maximal form $\quad\quad$ 1-3 mixing

Charged-lepton sector

$$m_E \sim \begin{pmatrix} y_e\langle\phi_{E1}\rangle & y_e\langle\phi_{E3}\rangle & y_e\langle\phi_{E2}\rangle \\ y_\mu\langle\phi_{E2}\rangle & y_\mu\langle\phi_{E1}\rangle & y_\mu\langle\phi_{E3}\rangle \\ y_\tau\langle\phi_{E3}\rangle & y_\tau\langle\phi_{E2}\rangle & y_\tau\langle\phi_{E1}\rangle \end{pmatrix} \qquad \langle\phi_{E2}\rangle = \langle\phi_{E3}\rangle = 0$$

$\mathbf{3}_L \times \mathbf{1}_R(\mathbf{1}_R', \mathbf{1}_R'') \times \mathbf{3}_{\text{flavon}} \to \mathbf{1}$ Assumptions

Diagonal mass matrix of charged-leptons!

So many bosons are introduced and vacuum alignments are assumed…

"Can we reduce the number of scalar?"

=>If only the SM Higgs is okay,

any vacuum alignments are not needed!

Modular flavor symmetry can resolve this issue!!!

Recent papers on modular symmetries

F.Feruglio:1706.08749; J.C.Criado, F.Feruglio:1807.01125,
T.Kobayashi,N.Omoto,Y.Shimizu, K.Takagi,M.T,T.H.Tatsuishi:1808.03012,
H.O., M. Tanimoto: 1812.09677 ; T. Nomura, H. O.: 1904.03937,
H.O., M. Tanimoto :1905.13421, F. J. de Anda, S. F. King and E. Perdomo:1812.05620,
P. P. Novichkov, S. T. Petcov and M. Tanimoto:1812.11289, T. Nomura and H. O.:1906.03927, G. J. Ding, S. F. King and X. G. Liu:1907.11714, H. O. and Y. Orikasa:1907.13520, T. Nomura, H. O and O. Popov:1908.07457, T. Kobayashi, Y. Shimizu, K. Takagi, M. Tanimoto and T. H. Tatsuishi:1909.05139,T. Asaka, Y. Heo, T. H. Tatsuishi and T. Yoshida:1909.06520, G. J. Ding, S. F. King, X. G. Liu and J. N. Lu:1910.03460p-ph], D. Zhang:1910.07869(Texture); H.O. T. Nomura, S.~Patra: 1912.00379,T.~Kobayashi, T. Nomura and T. Shimomura: 1912.00637, X.~Wang:X.~Wang:2020.115105, S.J.D.King, S.F.King:2002.00969, M. Abbas:2002.01929, H.~O. and Y.~Shoji:2003.13219, H.~O and M.~Tanimoto:2005.00775, M.~K.~Behera, S.~Mishra, S.~Singirala and R.~Mohanta:2007.00545, M.~K.~Behera, S.~Singirala, S.~Mishra and R.~Mohanta:2009.01806, T.~Nomura and H.O.:2007.15459, T.~Asaka, Y.~Heo and T.~Yoshida:2009.12120, H.Okada, M. Tanimoto:2009.14242, K.I.~Nagao and H.O.:2010.03348, J.N. Lu, etc.1912.07573(Double covering of A4), **H.O., M. Tanimoto :2012.01688, H.O. etc. (2012.11156),** C.Y. Yao, etc:2012.13390, P. Chen, etc,(2101.12724),
M. Abbas(Published in: Phys.Atom.Nucl. 83 (2020) 5, 764-769) ,… …
(more than 50 papers since 2017) for **A_4 or A'_4**

The other feature of modular symmetry

Dark Matter(DM)

stability can be assured by

making good use of the degrees of freedom in a modular symmetry

“1.modular weight”

and/or

“2.residual symmetries at fixed points”;

↓

Additional symmetry (Z_2) is not needed!

Mechanism 1.

$Y_r^{(k)}\bar{L}HN$ k is even ➡ Modular wight of $\bar{L}HN$ is even

We can assign Z_2 parity:
Fields with odd modular weight$\rightarrow Z_2$ odd
Fields with even modular weight$\rightarrow Z_2$ even

Ex.1

	Fermions			Bosons	
	$(\bar{L}_{L_e}, \bar{L}_{L_\mu}, \bar{L}_{L_\tau})$	$(e_{R_e}, e_{R_\mu}, e_{R_\tau})$	N_R	H	η^*
$SU(2)_L$	$\mathbf{2}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{2}$	$\mathbf{2}$
$U(1)_Y$	$\frac{1}{2}$	-1	0	$\frac{1}{2}$	$-\frac{1}{2}$
A_4	$1, 1', 1''$	$1, 1'', 1'$	3	1	1
$-k$	0	0	-1	0	-3

TABLE I: Fermionic and bosonic field content of the model and their charge assignments under $SU(2)_L \times U(1)_Y \times A_4$ in the lepton and boson sector, where $-k$ is the number of modular weight and the quark sector is the same as the SM.

A modular A4 symmetic scotogenic model:
T. Nomura, H.O., O. Popov, plb, 1908.07457

Dirac term is not allowed by modular weight
=> Predictvie Ma model is realized!

Mechanism 2.

$Y_r^{(k)}$ are zero at fixed points at τ=(i,ω,i∞)

↓

DM can be stable at these points.

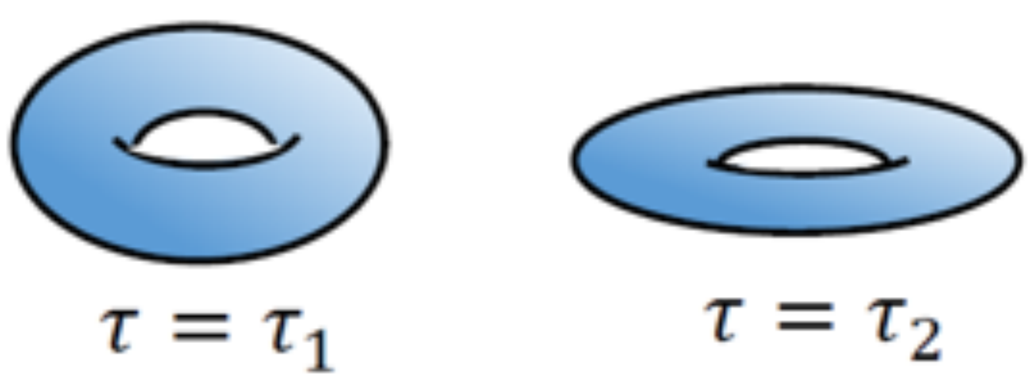
We will see this in details !!

Dark matter stability at fixed points in a A modular A4 symmetry:
T. Kobayashi, H.O., Y. Orikasa, 2111.05674

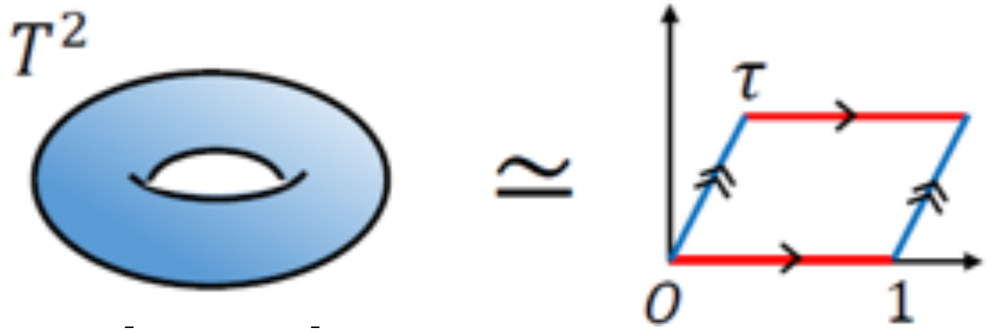
2. Modular Group

The superstring theory
on certain compactifications lead to
non-Abelian finite groups.

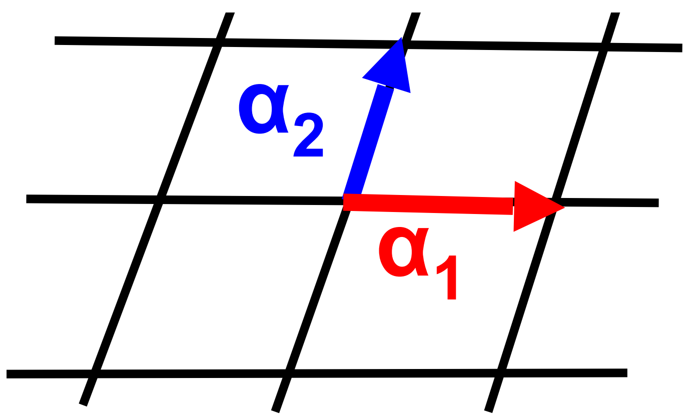
=>Two dimensional torus compactification leads to Modular symmetry,
which includes S_3, A_4, S_4, A_5 as its congruence subgroup.



The shape of torus is characterized by modulus τ, and different value of τ realizes the different shape of torus.



Torus can be described by a lattice on C-plane!



Identifying,

$(x,y) \sim (x,y) + n_1 \alpha_1 + n_2 \alpha_2$

The lattice is spanned by two vectors:

$\alpha_1 = 2\pi R$ and $\alpha_2 = 2\pi R T$ $\longrightarrow$ $T = \alpha_2 / \alpha_1$

Modular transformation

The same lattice is spanned by other bases under the transformation

$$\boxed{\begin{pmatrix} \alpha_2' \\ \alpha_1' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \alpha_2 \\ \alpha_1 \end{pmatrix}}$$

ad-bc=1

a,b,c,d are integer SL(2,Z)

Then, τ is transformed as follows:

$$\boxed{\tau \longrightarrow \tau' = \frac{a\tau + b}{c\tau + d}}$$

The modular transformation does not change the lattice(=shape of torus)!

The modular transformation is generated by S and T .

$$\boxed{\tau \longrightarrow \tau' = \frac{a\tau + b}{c\tau + d}}$$

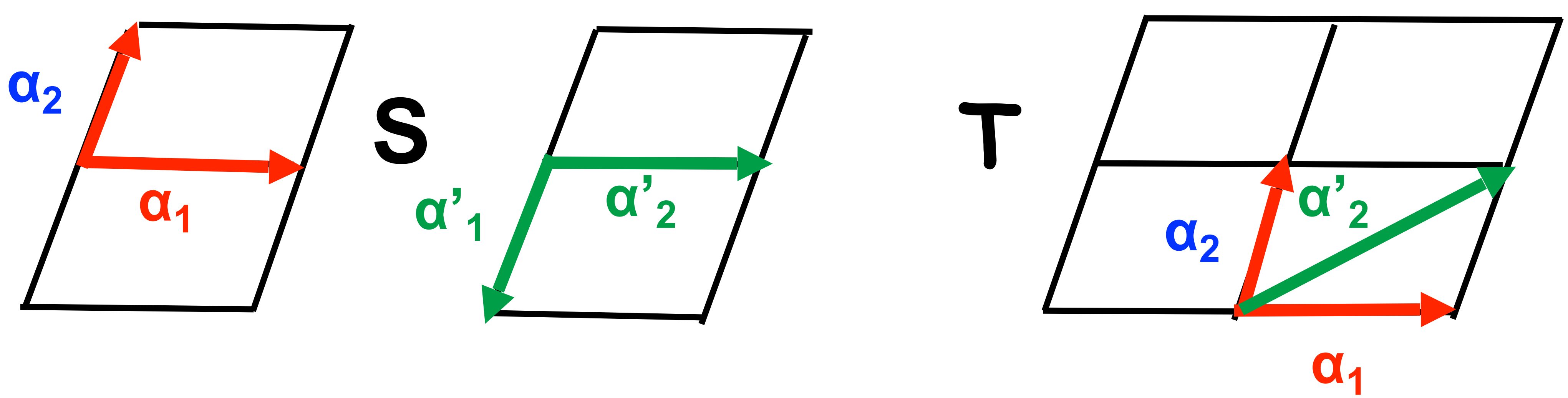
translation

$$S : \tau \longrightarrow -\frac{1}{\tau} \qquad T : \tau \longrightarrow \tau + 1$$

Matrix Form

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}^{t}$$

Lattice Form



$$S : \tau \longrightarrow -\frac{1}{\tau},$$
$$T : \tau \longrightarrow \tau + 1.$$

$$S^2 = 1, \qquad (ST)^3 = 1.$$

generate infinite discrete group

Modular group

Modular group

$$\Gamma \simeq \{S, T | S^2 = \mathbb{I}, (ST)^3 = \mathbb{I}\}$$

Point1.

Modular group has interesting finite subgroups !

If we impose T^N=1(congruence condition),

$$\bar{\Gamma}(N) \simeq \{S, T | S^2 = \mathbb{I}, (ST)^3 = \mathbb{I}, T^N = \mathbb{I}\}$$

$$\Gamma(N) \equiv \Gamma / \bar{\Gamma}(N)$$

$$\Gamma(2) \simeq S_3,\ \Gamma(3) \simeq A_4,\ \Gamma(4) \simeq S_4, \text{ and } \Gamma(5) \simeq A_5$$

Point2.
The modular group has recovered symmetries at fixed points; $\tau = i, \omega, i\infty$.

$\tau = i$

It is invariant under S. $\longrightarrow$ $Z_2(1,S)$ symmetry is recovered!

$\tau = \omega$

It is invariant under ST and (ST)(ST).

$\longrightarrow$ $Z_3(1,ST,(ST)(ST))$ symmetry is recovered!

$\tau = i\infty$

It is invariant under T, TT, TTT,…………T^N.

$\longrightarrow$ $Z_N(1,T,TT, \ldots T\hat{}N)$ symmetry is recovered!

#.N=3 is isomorphic to A_4 group.

How to construct the Yukawa coupling of modular A4 group?

$\mathcal{L}_{\text{eff}} \in f(\tau)\phi_1\phi_2\cdots\phi_n$ $\qquad$ $f(\tau), \phi_i$: non-trivial rep. of $\Gamma(N)$

$$\phi^{(I)} \to (c\tau + d)^{-k_I}\rho^{(I)}(\gamma)\phi^{(I)}$$

Modular weight $\qquad$ Representation matrix

$$f(\tau) \to f(\gamma\tau) = (c\tau + d)^k f(\tau)\ \rho(\gamma)$$

When k=sum[kI,{i,1,n}], and $\rho(\gamma)\times\rho$^(1)$\times\cdots\times\rho$^(n)=singlet, then it is invariant under the modular A4 group.

Mathematically, the minimum representation starts at triplet with modular weight of 2.

$$\mathbf{Y_3}^{(2)} = \begin{pmatrix} Y_1 \\ Y_2 \\ Y_3 \end{pmatrix}$$

But, the way of construction is not unique and too technical.
Thus, let me skip this process.

Result:

$$\begin{aligned}
Y_1(\tau) &= \frac{i}{2\pi}\left(\frac{\eta'(\tau/3)}{\eta(\tau/3)} + \frac{\eta'((\tau+1)/3)}{\eta((\tau+1)/3)} + \frac{\eta'((\tau+2)/3)}{\eta((\tau+2)/3)} - \frac{27\eta'(3\tau)}{\eta(3\tau)}\right), \\
Y_2(\tau) &= \frac{-i}{\pi}\left(\frac{\eta'(\tau/3)}{\eta(\tau/3)} + \omega^2\frac{\eta'((\tau+1)/3)}{\eta((\tau+1)/3)} + \omega\frac{\eta'((\tau+2)/3)}{\eta((\tau+2)/3)}\right), \\
Y_3(\tau) &= \frac{-i}{\pi}\left(\frac{\eta'(\tau/3)}{\eta(\tau/3)} + \omega\frac{\eta'((\tau+1)/3)}{\eta((\tau+1)/3)} + \omega^2\frac{\eta'((\tau+2)/3)}{\eta((\tau+2)/3)}\right),
\end{aligned}$$

$$\eta(\tau) = q^{1/24}\prod_{n=1}^{\infty}(1-q^n)$$ Dedekind eta-function

$$\eta(-1/\tau) = \sqrt{-i\tau}\,\eta(\tau), \qquad \eta(\tau+1) = e^{i\pi/12}\eta(\tau)$$

$$\begin{aligned}
Y_1(\tau) &= 1 + 12q + 36q^2 + 12q^3 + \cdots, \\
Y_2(\tau) &= -6q^{1/3}(1 + 7q + 8q^2 + \cdots), \\
Y_3(\tau) &= -18q^{2/3}(1 + 2q + 5q^2 + \cdots).
\end{aligned}$$

$$q = e^{2\pi i\tau}$$

$|q| \ll 1$

$$Y_2^2 + 2Y_1Y_3 = 0$$

Once this coupling is found, the other terms are automatically constructed by using multiplication rules of A4 group.

A multiplication rule of A4 group:

$$\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}_{\mathbf{3}} \otimes \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}_{\mathbf{3}} = (a_1b_1 + a_2b_3 + a_3b_2)_{\mathbf{1}} \oplus (a_3b_3 + a_1b_2 + a_2b_1)_{\mathbf{1'}}$$

$$\oplus (a_2b_2 + a_1b_3 + a_3b_1)_{\mathbf{1''}}$$

$$\oplus \boxed{\frac{1}{3}\begin{pmatrix} 2a_1b_1 - a_2b_3 - a_3b_2 \\ 2a_3b_3 - a_1b_2 - a_2b_1 \\ 2a_2b_2 - a_1b_3 - a_3b_1 \end{pmatrix}_{\mathbf{3}}} \oplus \boxed{\frac{1}{2}\begin{pmatrix} a_2b_3 - a_3b_2 \\ a_1b_2 - a_2b_1 \\ a_1b_3 - a_3b_1 \end{pmatrix}_{\mathbf{3}}}.$$

<u>(a1,a2,a3) is replaced by (Y1(τ),Y2(τ),Y3(τ)).</u>

Eg.

k=4:

$$\mathbf{Y}_{1}^{(4)} = Y_1^2 + 2Y_2Y_3\ , \quad \mathbf{Y}_{1'}^{(4)} = Y_3^2 + 2Y_1Y_2\ , \quad \mathbf{Y}_{1''}^{(4)} = Y_2^2 + 2Y_1Y_3 = 0\ ,$$

$$\mathbf{Y}_{3}^{(4)} = \begin{pmatrix} Y_1^2 - Y_2Y_3 \\ Y_3^2 - Y_1Y_2 \\ Y_2^2 - Y_1Y_3 \end{pmatrix}\ ,$$

k=6:

$$\mathbf{Y}_{1}^{(6)} = Y_1^3 + Y_2^3 + Y_3^3 - 3Y_1Y_2Y_3\ ,$$

$$\boxed{\mathbf{Y}_{3}^{(6)} \equiv \begin{pmatrix} Y_1^{(6)} \\ Y_2^{(6)} \\ Y_3^{(6)} \end{pmatrix} = \begin{pmatrix} Y_1^3 + 2Y_1Y_2Y_3 \\ Y_1^2Y_2 + 2Y_2^2Y_3 \\ Y_1^2Y_3 + 2Y_3^2Y_2 \end{pmatrix}\ , \qquad \mathbf{Y}_{3'}^{(6)} \equiv \begin{pmatrix} Y_1'^{(6)} \\ Y_2'^{(6)} \\ Y_3'^{(6)} \end{pmatrix} = \begin{pmatrix} Y_3^3 + 2Y_1Y_2Y_3 \\ Y_3^2Y_1 + 2Y_1^2Y_2 \\ Y_3^2Y_2 + 2Y_2^2Y_1 \end{pmatrix}}$$

...

<u>**Dof for each of representation is given by k+1.**</u>

Especially, we find representations at fixed points to find DM candidate:

k	$\mathbf{r}$	$\tau = i$	$\tau = \omega$	$\tau = i\infty$
2	$\mathbf{3}$	$(1, 1-\sqrt{3}, -2+\sqrt{3})$	$(1, \omega, -\frac{1}{2}\omega^2)$	$(1,0,0)$
4	$\mathbf{3}$	$(1,1,1)$	$(1, -\frac{1}{2}\omega, \omega^2)$	$(1,0,0)$
	$\{\mathbf{1}, \mathbf{1}'\}$	$\{1, \quad 1\}$	$\{0, \quad 1\}$	$\{1, \quad 0\}$
6	$\mathbf{3}_1$	$(1, 1-\sqrt{3}, -2+\sqrt{3})$	$(0,0,0)$	$(1,0,0)$
	$\mathbf{3}_2$	$(1, 1-\sqrt{3}, 1+\sqrt{3})$	$(1, \omega, -\frac{1}{2}\omega^2)$	$(0,0,0)$
	$\mathbf{1}$	0	1	1
8	$\mathbf{3}_1$	$(1,1,1)$	$(0,0,0)$	$(1,0,0)$
	$\mathbf{3}_2$	$(1,1,1)$	$(1, -\frac{1}{2}\omega, \omega^2)$	$(0,0,0)$
	$\{\mathbf{1}, \mathbf{1}', \mathbf{1}''\}$	$\{1, \quad 1, \quad 1\}$	$\{0, \quad 0, \quad 1\}$	$\{1, \quad 0, \quad 0\}$
10	$\mathbf{3}_1$	$(1, 1-\sqrt{3}, -2+\sqrt{3})$	$(0,0,0)$	$(1,0,0)$
	$\mathbf{3}_2$	$(1, 1-\sqrt{3}, 1+\sqrt{3})$	$(0,0,0)$	$(0,0,0)$
	$\mathbf{3}_3$	$(1, 1-\sqrt{3}, -2+\sqrt{3})$	$(1, \omega, -\frac{1}{2}\omega^2)$	$(0,0,0)$
	$\{\mathbf{1}, \mathbf{1}'\}$	$\{0, \quad 0\}$	$\{0, \quad 1\}$	$\{1, \quad 0\}$

Triplet rep. would be difficult to identify the DM candidate for A4 modular group.

…Promising representations to provide DM.

3. A concrete model

Dark matter stability at fixed points in a A modular A4 symmetry:
T. Kobayashi, H.O., Y. Orikasa, 2111.05674

We consider canonical seesaw, focussing on τ=i.

	Fermions			Bosons
	$(\overline{L_{L_e}}, \overline{L_{L_\mu}}, \overline{L_{L_\tau}})$	(e_R, μ_R, τ_R)	$(N_{R_1}, N_{R_2}, N_{R_3})$	H
$SU(2)_L$	$\mathbf{2}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{2}$
$U(1)_Y$	$\frac{1}{2}$	-1	0	$\frac{1}{2}$
A_4	$(1, 1'', 1')$	$(1, 1', 1'')$	$(1, 1', 1'')$	1
$-k$	-4	0	$(-2, -4, -4)$	0

DM candidate!

Renormalizable Yukawa Lagrangian

$$
\begin{aligned}
-\mathcal{L} = &\, a_\ell Y_1^{(4)} \overline{L_{L_e}} H e_R + b_\ell Y_1^{(4)} \overline{L_{L_\mu}} H \mu_R + c_\ell Y_1^{(4)} \overline{L_{L_\tau}} H \tau_R \\
&+ a'_\ell Y_{1'}^{(4)} \overline{L_{L_e}} H \tau_R + b'_\ell Y_{1'}^{(4)} \overline{L_{L_\mu}} H e_R + c'_\ell Y_{1'}^{(4)} \overline{L_{L_\tau}} H \mu_R \\
&+ a_D Y_1^{(6)} \overline{L_{L_e}} \tilde{H} N_{R_1} + b_D Y_1^{(8)} \overline{L_{L_\mu}} \tilde{H} N_{R_2} + c_D Y_1^{(8)} \overline{L_{L_\tau}} \tilde{H} N_{R_3} \\
&+ a'_D Y_{1'}^{(8)} \overline{L_{L_\tau}} \tilde{H} N_{R_2} + b'_D Y_{1'}^{(8)} \overline{L_{L_e}} \tilde{H} N_{R_3} + a''_D Y_{1''}^{(8)} \overline{L_{L_\mu}} \tilde{H} N_{R_3} + b''_D Y_{1''}^{(8)} \overline{L_{L_e}} \tilde{H} N_{R_2} \\
&+ M_1 Y_1^{(4)} \overline{N_{R_1}^C} N_{R_1} + M_{23} Y_1^{(8)} \overline{N_{R_2}^C} N_{R_3} + M_2 Y_{1'}^{(8)} \overline{N_{R_2}^C} N_{R_2} + M_3 Y_{1''}^{(8)} \overline{N_{R_3}^C} N_{R_3} + \text{h.c.}.
\end{aligned}
$$

$\tau = i$, a_D term vanishes!

↓

N_{R1} does not couple to any other fields, it is stable!

How much can the Yukawa be deviated from τ=i???

DM is a quasi-stable particle that is longer than age of Universe~10^17 sec.

Total decay width

$$\Gamma_X \simeq \frac{|Y|^2 m_X}{16\pi}\left(1-\frac{m_h^2}{m_X^2}\right)^2 .$$

$$\tau_X = \Gamma_X^{-1} \simeq 3.31 \times 10^{-26} \times \frac{1}{|Y|^2}\left(\frac{1\ \mathrm{TeV}}{m_X}\right)\left(1-\frac{m_h^2}{m_X^2}\right)^{-2} \mathrm{sec}.$$

|Y| should be less than 10^(-21) to satisfy the age of Universe!

Deviation would not be appropriate to include DM!

How about the neutrino mass matrix at exact value at τ=i?

Dirac mass

$$m_D = \frac{v}{\sqrt{2}} \begin{pmatrix} 0 & b_D'' \, Y_{1''}^{(8)} & b_D' \, Y_{1'}^{(8)} \\ 0 & b_D \, Y_1^{(8)} & a_D'' \, Y_{1''}^{(8)} \\ 0 & a_D' \, Y_{1'}^{(8)} & c_D \, Y_1^{(8)} \end{pmatrix}$$

Majorana mass

$$M_N = \frac{1}{\sqrt{2}} \begin{pmatrix} M_1 \, Y_1^{(4)} & 0 & 0 \\ 0 & M_2 \, Y_{1'}^{(8)} & M_{23} \, Y_1^{(8)} \\ 0 & M_{23} \, Y_1^{(8)} & M_3 \, Y_{1''}^{(8)} \end{pmatrix}$$

Neutrino mass

Rank 2 neutrino mass matrix!

$$m_\nu \approx -m_D M_N^{-1} m_D^T$$

Mass eigen values : $m_1 = 0, m_2 = \sqrt{\Delta m_{sol}^2}, m_3 = \sqrt{\Delta m_{atm}^2}$ (NH)

$$m_1 = \sqrt{\Delta m_{atm}^2 - \Delta m_{sol}^2}, m_2 = \sqrt{\Delta m_{atm}^2}, m_3 = 0 \text{ (IH)}$$

Hierarchical mass order for NH

Degenerate mass order for IH

NH

$$\frac{\sqrt{2}}{v} m_e: \begin{pmatrix} 0.0586 & 0 & 0.00385 + 0.00943\, i \\ -0.00718 + 0.00193\, i & 0.999 & 0 \\ 0 & -0.00187 & 0.000253 \end{pmatrix}$$

$$\frac{\sqrt{2}}{v} m_D: \begin{pmatrix} 0 & 0.0161 & 0.0148 - 0.589\, i \\ 0 & -0.324 & 0.00542 + 0.00912\, i \\ 0 & 0.158 - 0.0659\, i & 0.173 \end{pmatrix}$$

$$\frac{\sqrt{2}}{M_1} M_N: \begin{pmatrix} -0.00370 & 0 & 0 \\ 0 & -0.00700 - 0.0962\, i & 0.0808 - 0.137\, i \\ 0 & 0.0808 - 0.137\, i & 0.0224 + 0.00388\, i \end{pmatrix}$$

$M_1 = 2.4\ 10^{13} GeV$

$$\Rightarrow$$

$$\Delta m^2_{atm} = 2.47\ 10^{-3} eV^2$$
$$\Delta m^2_{sol} = 7.94\ 10^{-5} eV^2$$
$$sin\theta_{12} = 0.520$$
$$sin\theta_{23} = 0.759$$
$$sin\theta_{13} = 0.146$$
$$\delta_{CP} = 211^o$$
$$\alpha_{21} = 128^o$$

with in 3σ.

IH

$$\frac{\sqrt{2}}{v} m_e : \begin{pmatrix} 0.0594 & 0 & -0.00258 + 0.000803\, i \\ -0.0242 + 0.00188\, i & 0.999 & 0 \\ 0 & 0.0265 + 0.00177\, i & 0.000259 \end{pmatrix}$$

$$\frac{\sqrt{2}}{v} m_D : \begin{pmatrix} 0 & -0.185 & 0.000992 - 0.0428\, i \\ 0 & -0.201 & 0.0278 + 0.00309\, i \\ 0 & -0.000833 & -0.238 \end{pmatrix}$$

$$\frac{\sqrt{2}}{M_1} M_N : \begin{pmatrix} 0.00390 & 0 & 0 \\ 0 & 0.0592 - 0.0695\, i & -0.0116 + 0.00427\, i \\ 0 & -0.0116 + 0.00427\, i & 0.0134 - 0.0735\, i \end{pmatrix}$$

$$\Rightarrow$$

$$\Delta m^2_{atm} = 2.43\ 10^{-3} eV^2$$
$$\Delta m^2_{sol} = 7.84\ 10^{-5} eV^2$$
$$sin\theta_{12} = 0.561$$
$$sin\theta_{23} = 0.735$$
$$sin\theta_{13} = 0.153$$
$$\delta_{CP} = 284^o$$
$$\alpha_{21} = 352^o$$

with in 3σ.

$$M_1 = 4.3\ 10^{12} GeV$$

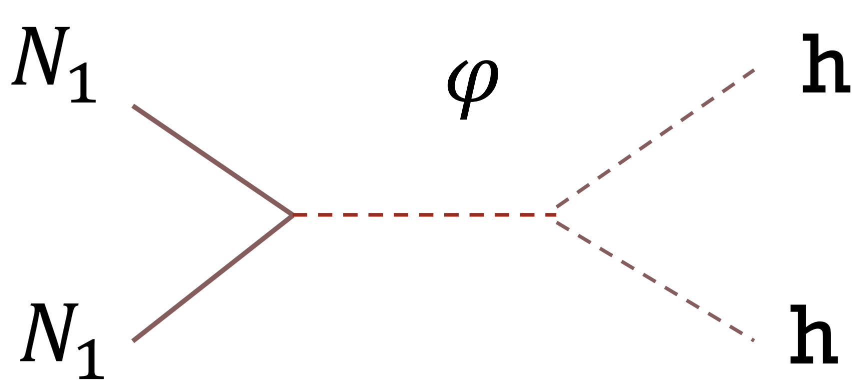
How to explain the relic density of DM?

A simple way is to extend the gauged U(1) symmetry.

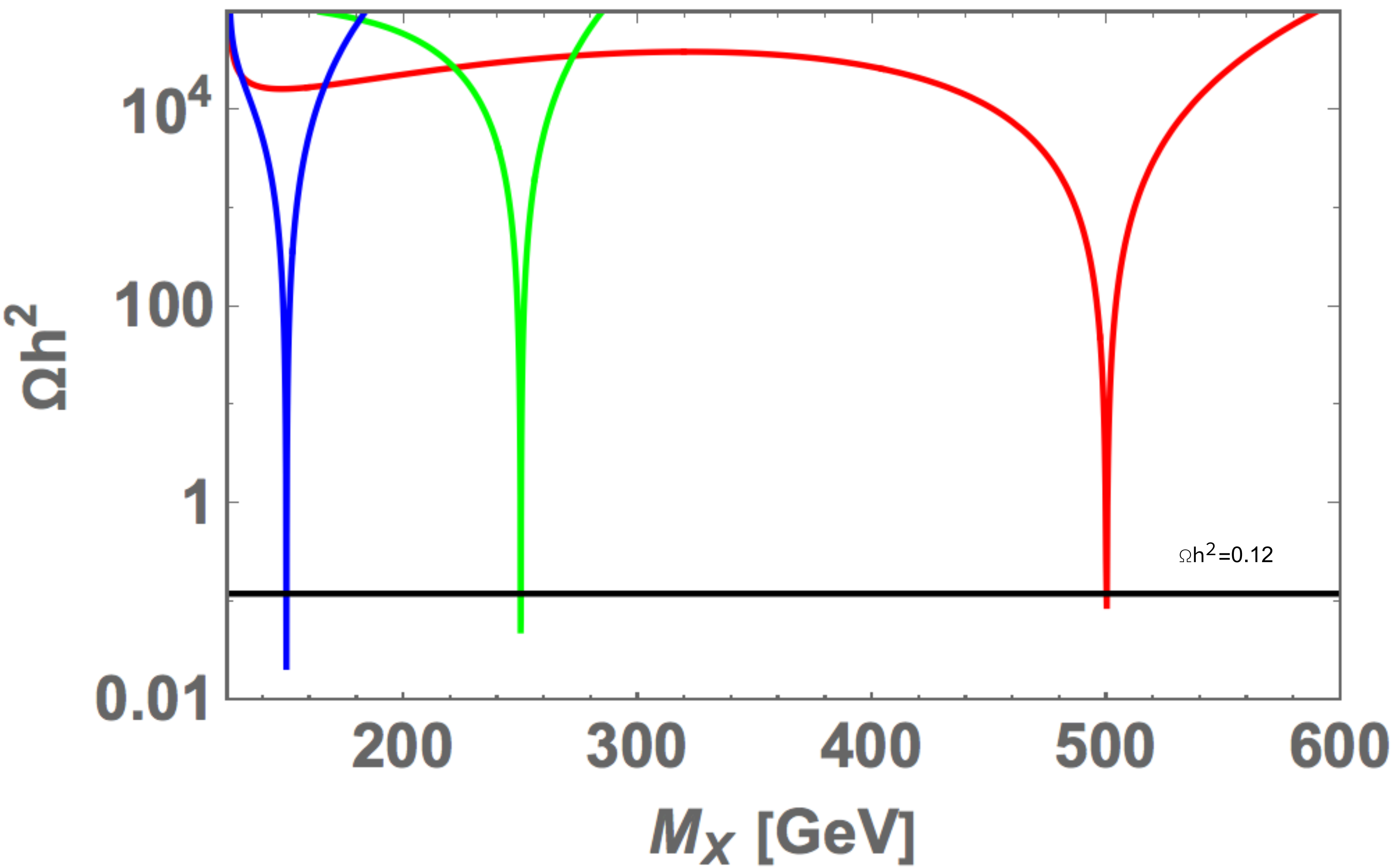
The difference appears to only the Majorana mass terms; model does not spoil.

$\tau = i:$

$$y_{M_1}\varphi Y_1^{(4)}(i)\overline{N_{R_1}^C}N_{R_1} + y_{M_{23}}\varphi Y_1^{(8)}(i)\overline{N_{R_2}^C}N_{R_3} + y_{M_2}\varphi Y_{1'}^{(8)}(i)\overline{N_{R_2}^C}N_{R_2} + y_{M_3}\varphi Y_{1''}^{(8)}(i)\overline{N_{R_3}^C}N_{R_3} + \mathrm{h.c.}...$$



The solutions are found at nearby resonant points.



4. Summary

- Several modular forms with certain rep. become zero at fixed points, since a symmetry is recovered.

- We successfully constructed a model at $\tau=i$ that explains DM as well as reproduce neutrino oscillation data within 3σ.

Next task is to provide predictions in the neutrino sector.

S4 or the other modular groups(with doublet) would be promising...

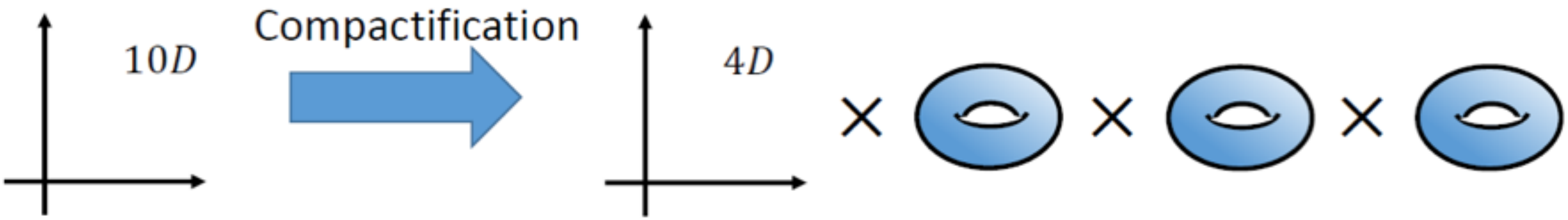
Many thanks!

Back up!

Superstring theory 10D
Our universe is 4D

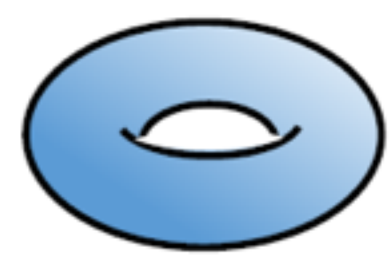
→ **The extra 6D should be compactified.**

Torus compactification



We get 4D effective Lagrangian by integrating out over 6D.

$$S = \int d^4x d^6y\, \mathcal{L}_{10D} \rightarrow \int d^4x\, \mathcal{L}_{\text{eff}}$$

→ $\mathcal{L}_{\text{eff}}$ depends on the structure of 

- $4D$ effective theory depends on internal space

Once the Yukawa structure of triplet with 2 modular weight is obtained, any kinds of Yukawa structures are straightforwardly found via multiplication rules of A_4.

$$\mathbf{Y}_{\mathbf{3}}^{(2)} = \begin{pmatrix} Y_1(\tau) \\ Y_2(\tau) \\ Y_3(\tau) \end{pmatrix}$$

k=4

$$\mathbf{Y}_{\mathbf{1}}^{(4)} = Y_1^2 + 2Y_2Y_3 \,, \quad \mathbf{Y}_{\mathbf{1'}}^{(4)} = Y_3^2 + 2Y_1Y_2 \,, \quad \boxed{\mathbf{Y}_{\mathbf{1''}}^{(4)} = Y_2^2 + 2Y_1Y_3 = 0 \,,}$$

$|q| \ll 1$

$$\mathbf{Y}_{\mathbf{3}}^{(4)} = \begin{pmatrix} Y_1^2 - Y_2Y_3 \\ Y_3^2 - Y_1Y_2 \\ Y_2^2 - Y_1Y_3 \end{pmatrix} \,,$$

k=6

$$\mathbf{Y}_{\mathbf{1}}^{(6)} = Y_1^3 + Y_2^3 + Y_3^3 - 3Y_1Y_2Y_3 \,,$$

$$\mathbf{Y}_{\mathbf{3}}^{(6)} \equiv \begin{pmatrix} Y_1^{(6)} \\ Y_2^{(6)} \\ Y_3^{(6)} \end{pmatrix} = \begin{pmatrix} Y_1^3 + 2Y_1Y_2Y_3 \\ Y_1^2Y_2 + 2Y_2^2Y_3 \\ Y_1^2Y_3 + 2Y_3^2Y_2 \end{pmatrix} \,, \qquad \mathbf{Y}_{\mathbf{3'}}^{(6)} \equiv \begin{pmatrix} Y_1'^{(6)} \\ Y_2'^{(6)} \\ Y_3'^{(6)} \end{pmatrix} = \begin{pmatrix} Y_3^3 + 2Y_1Y_2Y_3 \\ Y_3^2Y_1 + 2Y_1^2Y_2 \\ Y_3^2Y_2 + 2Y_2^2Y_1 \end{pmatrix} \,.$$

Singlets start from # of modular weight 4.

We can consider effective theories with Γ(N) symmetry.

$\mathcal{L}_{\text{eff}} \in f(\tau)\phi_1\phi_2\cdots\phi_n$ $f(\tau), \phi_i$: non-trivial rep. of $\Gamma(N)$

In some cases, explicit form of function f (τ) have been obtained.

Famous modular function : Dedekind eta-function

$$\eta(\tau) = q^{1/24}\prod_{n=1}^{\infty}(1-q^n) \qquad q = e^{2\pi i\tau}$$

$$\eta(-1/\tau) = \boxed{\sqrt{-i\tau}\,\eta(\tau)}, \qquad \eta(\tau+1) = e^{i\pi/12}\eta(\tau)$$

So called **Modular weight 1/2**

Modular transformation of fields:

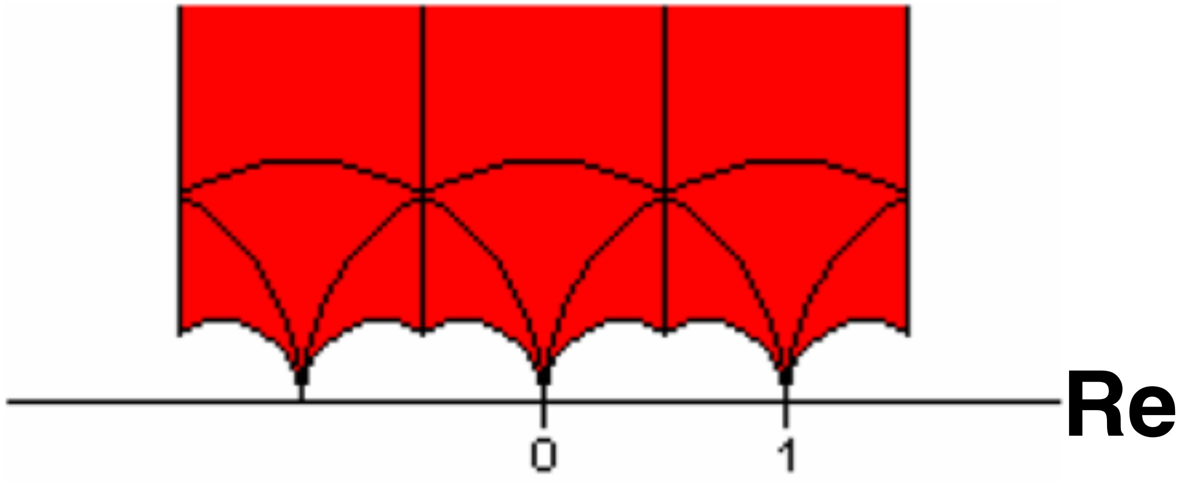
$$\phi^{(I)} \to (c\tau+d)^{-k_I}\rho^{(I)}(\gamma)\phi^{(I)}$$

Modular weight Representation matrix

Take $T^3=1$ $\Gamma(3) \simeq A_4$ group

N	g	$d_{2k}(\Gamma(N))$	μ_N	Γ_N
2	0	$k+1$	6	S_3
3	0	$\boxed{2k+1}$	12	$\boxed{A_4}$
4	0	$4k+1$	24	S_4
5	0	$10k+1$	60	A_5
6	1	$12k$	72	
7	3	$28k-2$	168	

2k is weight



Fundamental domain of T

There are 3 linearly independent modular forms for 2k=2 (weight 2)

Dimension $d_{2k}(\Gamma(3))=2k+1$

Triplet !

General remarks: Any singlets start from # of modular weight 4.

How to find A_4 triplet modular functions.

Prepare 4 Dedekind eta-functions as Modular functions

$$\eta(-1/\tau) = \sqrt{-i\tau}\,\eta(\tau), \qquad \eta(\tau+1) = e^{i\pi/12}\eta(\tau)$$

$$\eta(3\tau) \to \sqrt{\frac{-i\tau}{3}}\,\eta(\tau/3),$$
$$\eta(\tau/3) \to \sqrt{-i3\tau}\,\eta(3\tau),$$
$$\eta((\tau+1)/3) \to e^{-i\pi/12}\sqrt{-i\tau}\,\eta((\tau+2)/3),$$
$$\eta((\tau+2)/3) \to e^{i\pi/12}\sqrt{-i\tau}\,\eta((\tau+1)/3).$$

S : τ → -1/τ

$$\eta(3\tau) \to e^{i\pi/4}\eta(3\tau),$$
$$\eta(\tau/3) \to \eta((\tau+1)/3),$$
$$\eta((\tau+1)/3) \to \eta((\tau+2)/3),$$
$$\eta((\tau+2)/3) \to e^{i\pi/12}\eta(\tau/3),$$

T : τ → τ+1

Modular function with weight 2 by using Dedekind eta-function

$$Y(\alpha,\beta,\gamma,\delta|\tau) = \frac{d}{d\tau}\left(\alpha \log\eta(\tau/3) + \beta\log\eta((\tau+1)/3) + \gamma\log\eta((\tau+2)/3) + \delta\log\eta(3\tau)\right)$$

$$\alpha+\beta+\gamma+\delta=0$$

$$S:\tau \longrightarrow -\frac{1}{\tau}, \qquad S:\quad Y(\alpha,\beta,\gamma,\delta|\tau) \to \tau^2 Y(\delta,\gamma,\beta,\alpha|\tau),$$

$$T:\tau \longrightarrow \tau+1. \qquad T:\quad Y(\alpha,\beta,\gamma,\delta|\tau) \to Y(\gamma,\alpha,\beta,\delta|\tau).$$

In A_4 group, $T^3=1$

$$\rho(S) = \frac{1}{3}\begin{pmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{pmatrix}, \qquad \rho(T) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{pmatrix},$$

F. Feruglio, arXiv:1706.08749

A_4 triplet of modular function with weight 2

$$\begin{pmatrix} Y_1(-1/\tau) \\ Y_2(-1/\tau) \\ Y_3(-1/\tau) \end{pmatrix} = \tau^2 \rho(S) \begin{pmatrix} Y_1(\tau) \\ Y_2(\tau) \\ Y_3(\tau) \end{pmatrix}, \qquad \begin{pmatrix} Y_1(\tau+1) \\ Y_2(\tau+1) \\ Y_3(\tau+1) \end{pmatrix} = \rho(T) \begin{pmatrix} Y_1(\tau) \\ Y_2(\tau) \\ Y_3(\tau) \end{pmatrix}.$$

$$\begin{aligned}
Y_1(\tau) &= \frac{i}{2\pi}\left(\frac{\eta'(\tau/3)}{\eta(\tau/3)} + \frac{\eta'((\tau+1)/3)}{\eta((\tau+1)/3)} + \frac{\eta'((\tau+2)/3)}{\eta((\tau+2)/3)} - \frac{27\eta'(3\tau)}{\eta(3\tau)}\right), \\
Y_2(\tau) &= \frac{-i}{\pi}\left(\frac{\eta'(\tau/3)}{\eta(\tau/3)} + \omega^2\frac{\eta'((\tau+1)/3)}{\eta((\tau+1)/3)} + \omega\frac{\eta'((\tau+2)/3)}{\eta((\tau+2)/3)}\right), \\
Y_3(\tau) &= \frac{-i}{\pi}\left(\frac{\eta'(\tau/3)}{\eta(\tau/3)} + \omega\frac{\eta'((\tau+1)/3)}{\eta((\tau+1)/3)} + \omega^2\frac{\eta'((\tau+2)/3)}{\eta((\tau+2)/3)}\right),
\end{aligned}$$

$$\begin{aligned}
Y_1(\tau) &= 1 + 12q + 36q^2 + 12q^3 + \cdots, \\
Y_2(\tau) &= -6q^{1/3}(1 + 7q + 8q^2 + \cdots), \\
Y_3(\tau) &= -18q^{2/3}(1 + 2q + 5q^2 + \cdots).
\end{aligned}$$

$q = e^{2\pi i\tau}$

$|q| \ll 1$

$Y_2^2 + 2Y_1Y_3 = 0$